HINDU GEOMETRY

by

BIBHUTIBHUSAN DATTA AND AVADHESH NARAYAN SINGH

Revised by

KRIPA SHANKAR SHUKLA

Department of Mathematics and Astronomy, Lucknow University, Lucknow

(*Received 15 February 1979*)

CONTENTS

1. GENERAL SURVEY

Origin of Hindu Geometry

The Hindu Geometry originated in a very remote age in connection with the construction of the altars for the Vedic sacrifices. The sacrifices, as described in the Vedic literature of the Hindus, were of various kinds. The performance of some of them was obligatory upon every Vedic Hindu, and hence they were known as *Nitya* (or "obligatory", "indispensable"). Other sacrifices were to be performed each with the purpose of achieving some special object. Those who did not aim at the attainment of any such object had no need to perform any of them. These sacrifices were classed as *Kāmya* (or "optional", "intentional"). According to the strict injunctions of the Hindu scriptures, each sacrifice must be made in an altar of prescribed shape and size. It was emphasized that even a slight irregularity and variation in the form and size of the altar would nullify the object of the whole ritual and might even lead to an adverse effect. So the greatest care had to be taken to secure the right

shape and size of the altar. In this way there arose in ancient India problems of geometry and also of arithmetic and algebra. There were multitudes of altars. Let us take for instance the three primary ones, viz. the *Gārhapatya*, *Āhavanīya* and *Dakṣiṇa*, in which every Vedic Hindu had to offer sacrifices daily. The *Gārhapatya* altar was prescribed to be of the form of a square, according to one school, and of a circle according to another. The *Āhavanīya* altar had always to be square and the *Dakṣiṇa* altar semi-circular. But the area of each had to be the same and equal to one square *vyāma*[1]. So the construction of these three altars involved three geometrical operations : (i) to construct a square on a given straight line ; (ii) to circle a square and vice versa ; and (iii) to double a circle. The last problem is the same as the evaluation of the surd $\sqrt{2}$. Or it may be considered as a case of doubling a square and then circling it. There were altars of the shape of a falcon with straight or bent wings, of a square, an equilateral triangle, an isosceles trapezium, a circle, a wheel (with or without spokes), a tortoise, a trough and of other complex forms all having the same area. Again at the second and each subsequent construction of an altar, it was necessary to increase its size by a specified amount, usually one square *puruṣa*,[2] but the shape was always kept similar to that of the first construction. Thus there arose problems of equivalent areas and transformation of areas. The Vedic geometers also treated problems of 'application of areas'.

Different Early Schools of Geometry

In the course of time, Hindu geometry grew beyond its original sacrificial purpose or the bounds of practical utility and began to be cultivated as a science for its own sake. This happened in the Vedic age when different schools of geometry were founded. More notable ones amongst them were the schools of Baudhāyana, Āpastamba and Kātyāyana. Though the geometrical propositions treated in all of them were almost the same, and there were many things common in the methods of their solution, still there were other things to distinguish one school from another. Even in the solution of elementary propositions such as the construction of a square, rectangle or an isosceles trapezium, different schools had preferential liking for differential methods. The difference appears most marked in the solution of the problems of the division of figures. The large altars, of which the fundamental one was of the shape of a falcon, had to be built with 200 bricks. Geometrically, it was a case of division of a figure into 200 parts. We have described before how the different Vedic Schools of Geometry did this in different ways.

---

1. 1 *Vyāma*=96 *aṅgulis* (or "finger breadths")=2 yards.
2. 1 *puruṣa*=120 *aṅgulis*=$2\frac{1}{2}$ yards.

Intuitive and Demonstrative Geometry

Early Hindu geometers did not describe proofs of the propositions discovered by them. Only the bare results were recorded and those too in a language as concise as possible, sometimes even to the fault of ellipticity. This was, of course, in keeping with the characteristic of the Hindu race and was manifested in all their early works. Indeed the character of all the sciences of all the early nations is found to be more or less intuitive. Still the Vedic Geometry, as found in the manuals of the *Śulba*, was not wholly intuitional without any semblance of demonstration. In fact we find a kind of proof in case of certain propositions of the *Śulba*. For instance, how to find the area of a trapezium, has been demonstrated by Āpastamba in the course of the mensuration of the *Mahāvedi* which is of the shape of an isosceles trapezium whose altitude, face and base are respectively 36, 24 and 30 *padas* (or *prakramas*). He says :

"The *Mahāvedi* measures (in area) one thousand less twenty-eight (square) *padas*. Draw a straight line from the south-eastern corner of the *vedi* to a point 12 *padas* towards the south-western corner. Place the portion thus cut off on the other (i. e. the northern) side of the *vedi* after inverting it. It (the *Mahāvedi*) will then become a rectangle. After that construction the area will be apparent."[1]

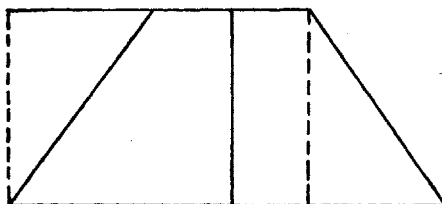


FIG. 1

After the general enunciation of the theorem of the square of the diagonal, the so-called Pythagorean theorem, Baudhāyana observes that the truth of it will be "realised" in case of certain rational rectangles enumerated. This is an attempt for a kind of demonstration. After describing the constructions necessary in a proposition, the early Hindu geometers are found to have remarked *sa samādhiḥ* (or "This is the construction"). The significance of such an observation is obvious. It emphasizes that the construction which was required to be made, has thus been actually made, and indeed corresponds to the expression *Quod Erat Faciendum* (or "what was required to do") occurring at the end of a proposition of Euclid's *Elements*. Further it discloses a rational and demonstrative attitude of the mind of the early Hindu geometers[2].

1. *Āpastamba Śulba*, v. 7.
2. See Datta, B., *The Science of the Śulba*, pp. 50f.

Post-Vedic Geometry

The Hindu geometry which started in a brilliant way not only by going much in advance of the ancient Egyptian or Chinese geometry but also by anticipating some of the notable discoveries of the posterior Greek geometry, did not make much progress in the post-Vedic period as it ought to have done. In this period there was renaissance of Hindu Mathematics.[1] But compared with arithmetic and algebra, geometry seems to have received little impetus for further development. It will not be true to think that the study of geometry was completely neglected by the Hindus of the early renaissance period. On the other hand, it is found to have become widespread and came to be regarded as a part of general education of the people. In an early Jaina canonical work, composed *circa* 300 B.C., we find the remark, "Geometry is the lotus in Mathematics,...and the rest is inferior".[2] But it appears strange that we do not find evidence of much progress and improvement in geometry. The notable contributions of this period to geometry are, however, the discovery of the ellipse, elliptic cylinder, the value $\pi=\sqrt{10}$ and certain formulae for the mensuration of the segment of a circle. The value $\pi=\sqrt{10}$, though not a fairly accurate one, is an improvement upon the *Śulba* value. It occurs as early as in the *Sūryaprajñapti* (*c.* 500 B.C.).[3] The ellipse is called *Viṣama-cakravāla*, in contradistinction to *cakravāla*, meaning "circle" in the *Sūryaprajñapti*[4], and *parimaṇḍala* in the *Dhamma-saṅgani* (before 350 B.C.)[5] and *Bhagavatī-sūtra* (*c.* 300 B.C.).[6] In the last mentioned work its form has been described as the *yavamadhya-vṛtta-saṃsthāna* or "the circular figure resembling the middle (longitudinal section) of a barley corn".[7] It seems to have been known that the ellipse is symmetrical about its either axis.[8] The mention of the elliptic cylinder, called *ghana-parimaṇḍala* (or "solid ellipse") in contradistinction to *pratara-parimaṇḍala* ("plane ellipse") occurs in the *Bhagavatī-sūtra*.[9]

---

1. See Datta, Bibhutibhusan, "The Scope and Development of the Hindu *Gaṇita*", *Ind. His. Quart.*, V, (1929), pp. 479 ff. We have drawn here heavily on this article.
2. *Sūtrakṛtāṅga-sūtra*, 2nd *Śrutaskanda*, ch. 1, verse 154.
3. *Sūtra* 20.
4. *Sūtra* 19, 25, 100. See Weber, *Indische Studien*, X, p. 274.
5. Sec. 617.
6. *Sūtra* 726-7.
7. *Bhagavatī-sūtra*, *Sūtra* 725. Bhuddhaghosa (350) describes it as *Kukkuṭāṇḍa-saṃsthāna* (or "a figure of the shape of an egg of a hen") and the *Petavattu* commentary as the *āyatavṛtta* (or "the elongated circle").
8. Compare *Bhagavatī-sūtra*, *Sūtra* 726.
9. *Sūtra* 726.

Later Hindu Geometry

Later Hindu geometry consists mainly of some mensuration formulae and solution of certain rectilinear figures such as triangles and quadrilaterals of different varieties. In some of these the Hindus undoubtedly showed considerable proficiency and indeed they obtained some remarkable results, e.g. a new proof of the Pythagorean theorem, formulae for the area and diagonals of an inscribed convex quadrilateral and rational solution of triangles and cyclic quadrilaterals. But on the whole their geometry remained empirical. There were no definitions, no postulates, no axioms, no proofs of theorems, in short, no scientific treatment of the subject. It is perhaps noteworthy that the later Hindus included geometry in their treatises of arithmetic (*pāṭīgaṇita*) more particularly in the sections on *kṣetra* ("plane figures"), *Khāta* ("excavations"), *citi* ("piles of bricks"), *rāśi* ("maunds of grain") and *krākacika* ("saw"). The last four topics are pertaining to solid figures.

Euclid's Elements in India

Though Hindu geometry is not connected with Euclid's *Elements* in any way, whether directly or indirectly, it will be interesting to know when and how it came to India. The earliest attempt, as far as known, to introduce Euclid's *Elements* into India, in the garb of Sanskrit verses, was made by the eminent Persian mathematician and traveller, Al-Bīrūnī (b. 973). But that attempt did not succeed. With the establishment of Muhammadan supremacy in India towards the close of the twelfth century of the Christian era, Arabic and Persian works on mathematics began to be brought into this country. There were very likely amongst them Arabic versions of the *Elements*. King Firuz Shah Bahmani (1397-1422), we are informed by Ferishta, was used to hear on three days in a week, lectures on botany, geometry and logic.[1] A son of Daud Shah was very fond of *Taḥrīr-u-Uqlidas* (Euclid's *Elements*) and used to teach it regularly to his students.[2] Akbar (1575) included it into the course of study for the school boys.[3] In his *Ain-i-Akbarī*, Abul Fazl (1590) has referred to a few propositions of the *Elements* in a way which shows his thorough acquaintance with the work. The work, however, remained confined to the circle of Moslem schools in India. We do not find any trace of its influence in any work of a Hindu writer before the middle of the seventeenth century. In 1658 A. D. Kamalākara, the court-astronomer of the Emperor Jahangir of Delhi, wrote a voluminous

1. Law, N.N., *Promotion of Learning in India during Muhammadan Rule* (by Muhammadans), 1916, p. 84.
2. *Ibid*, p. 81, footnote 1.
3. Abul Fazl's *Aini-i-Akbari*, English translation by Blockmann, p. 279.

treatise on astronomy entitled *Siddhānta-tattva-viveka*. Certain passages in this work can be easily recognised to have been adapted from Euclid's Elements.[1] The first complete translation of the work in Sanskrit was made in 1718 A. D. under the title *Rekhāgaṇita* ("Mathematics of lines") by Samrāṭa Jagannātha, by the order of his patron King Jaya Siṁha of Jaipur. Another Sanskrit version is known as the *Siddhānta-Cuḍāmaṇi*. The author of this version is still unknown.

2. Hindu Names For Geometry

The Hindu name for the science of geometry has varied from time to time.[2] The earliest name was *Śulba*. It is at least as old as the *Śrauta-sūtra* of Āpastamba (*c*. 1000 B.C.). Geometry was then sometimes also called *Rajju*, as is evident from the opening aphorism of the *Śulba* of Kātyāyana, "I shall speak of the collection of (rules regarding) the *Rajju*". In the *Mānava Śulba* and *Maitrāyaṇīya Śulba* we get the name *Śulba-vijñāna* ("The Science of the *Śulba*") for the science of geometry. In the early canonical works of the Jainas (500-300 B.C.) the more common name for geometry is found to be *Rajju*.

The Sanskrit words *śulba* and *rajju* have the identical significance, which is ordinarily "a rope", "a cord". The word *śulba* (or *śulva*) is derived from the root *śulb* (or *śulv*) meaning" to measure" and hence its etymological significance is "measuring" or "act of measurement". From that it came to denote "a thing measured" and consequently "a line (or surface)" as well as "an instrument of measurement" or "the unit of measurement". Thus the terms *śulba* and *rajju* have four meanings : (1) mensuration—the act and process of measuring ; (2) line (or surface)—the result obtained by measuring ; (3) a measure—the instrument of measuring ; and (4) geometry—the science of measurement. Mention of a linear measure, called *rajju* is found in the *Āpastamba-śulba*, *Mānava-śulba*, *Arthaśāstra* of *Kauṭilya* and later on in the *Śilpa-śāstra*. In fact in ancient India, there were three kinds of measures—linear, superficial and voluminal—having the same epithet *rajju*. In the Jaina canonical works they are sometimes distinguished as *sūcī-rajju* ("needle-like or linear *rajju*"), *pratara-rajju* ("superficial *rajju*") and *ghana-rajju* ("cubic *rajju*"). In the *Arthaśāstra of Kauṭilya* the superficial unit of *rajju* is called *paridेśa* and the cubical unit *nivartana*. In the works on the *Śulba*, we find the use of the word *rajju* in the sense of a measuring tape as also of a line.

1. See *Siddhānta-tattva-viveka*, iii. 22 ff.

2. Datta, Bibhutibhusan, "Origin and history of the Hindu names for Geometry," *Quellen und. Studien z. Gesh. d. Math.*, Ab. B; Bd, 1, pp. 113-9.

In later times, geometry was called by the Hindus *Kṣetra-gaṇita* ("Mathematics of the *Kṣetra*".) This term appears in the *Gaṇita-sāra-saṃgraha* of Mahāvīra (850). In this work the term *kṣetra* denotes a plane figure. In the mathematical treatises of Brahmagupta (628), Śrīdhara (900) and Bhāskara II (1150), the section devoted to the treatment of plane figures is called *kṣetra-vyavahāra* ("Treatment of plane figures"). The epithet *kṣetra-gaṇita* occurs as early as the works of Siddhasena Gaṇi (550). There the term *kṣetra* has a wider connotation so as to include both areas and volumes. In the same significance it appears in the title of the Jaina cosmographical works called *kṣetra-samāsa*. We think that the term *kṣetra-gaṇita* had a wider connotation in the beginning so as to include the geometry of plane as well as solid figures. But in later times, when the two branches of geometry began to be treated separately, the old name was reserved only for the geometry of plane figures.

Jagannātha (1718) called his version of Euclid's *Elements* the *Rekhā-gaṇita* ("Mathematics of lines"). Bāpūdeva Śāstrī preferred the name *kṣetra-miti* ("Measurement of areas and volumes"). He seems to have intended an accurate translation of the Greek name, but it is less scientific. For the Greek science is indeed the geometry of lines, but not the geometry of areas and volumes. Jagannātha's epithet is more in keeping with the spirit of the Greek geometry. He had probably discarded the Greek epithet intentionally as it is a misnomer.

In some of the modern vernacular tongues of India, geometry is now more commonly known as *kṣetra-tattva* ("Principles of areas and volumes") or *Jyāmiti*. This latter term is highly interesting because it is very alike the Greek term "geometry", not only phonetically but also in significance, and at the same time it is not a hinduised Greek word. The word *Jyāmiti* is a compound of pure Sanskrit origin derived from *jyā*, meaning 'earth' and *miti*, meaning "measure". Hence its literal significance is "earth-measurement." It is thus clearly a translation of the Greek name.

One who was well versed in the science of geometry was called in ancient India as *Saṃkhyājña* ('the expert in numbers'), *Parimāṇajña* ('the expert in measuring'). *Sama-sūtra-nirañchaka* ('uniform-rope-stretcher'), *Śulba-vid* ('the expert in the *Śulba*') and *Śulba-paripṛcchaka* ('the inquirer into the *Śulba*'). In the *Śilpa-śāstra*, he is spoken of as the *sūtra-grāhī* or *sūtra-dhāra* ('rope-holder') and he is further described as an expert in alignment (*rekhājña*, lit. 'one who knows the line'). In the early *Pāli* literature we find the terms *rajjuka* and *rajju-grāhaka* ('rope-holder') for the king's land-surveyor. The first of these terms appears copiously in its various case-endings, in the inscriptions of the Emperor Aśoka (250 B. C.).

3. Technical Terms

Line

The history of a few technical terms of Hindu geometry will be considered here. There is no attempt to define those terms in any early work. Only in a work of the seventeenth century, *Siddhānta-tattva-viveka* of Kamalākara (1658), we come across some definitions but, as already stated, it was influenced by Euclid's *Elements*. The line is called in the *Śulba*, *rekhā* or *lekhā*, both the terms being identical as, according to the rules of Sanskrit grammar, the letters *r* and *l* can replace each other. In posterior geometry we, however, commonly meet with the term *rekhā* only. A straight line is distinguished with the help of the qualifying adjective *ṛju* or *sarala*, meaning "straight".

Rectilinear Figures

In Hindu geometry, we find two different systems of nomenclature for the rectilinear geometrical figures[1]. In one system the naming is according to the number of sides of the figures and the names are formed by juxtaposition of the number names with *bhuja*, meaning "arm", "side" ; e.g. *tribhuja* ('trilateral'), *catur-bhuja* ('quadrilateral'), *pañca-bhuja* ('pentalateral'), *ṣaḍ-bhuja* ('hexa-lateral'). In the other, the naming is based on the number of angles and corners in the figures, and the names are compounds of number names with *karṇa* or *koṇa*. The Sanskrit word *karṇa* means the ear. Applied to geometrical figures, it implies, the angle.[2] In the *Katyāyana Śulba*[3] (*c*. 500 B.C.), we find the terms *trikarṇa* ('triangle'), *pañca-karṇa* ('pentangle'). The word *karṇa* degenerated into *koṇa* in the *Prākṛta* languages.[4] So in the Ardha-Māgadhī work, *Sūryaprajñapti*[5] (*c*. 500 B. C.), we get *tri-koṇa* ('trigonon'), *catuṣkoṇa* ('tetragonon'), *pañca-koṇa* ('pentagon'), etc. These terms are, however, accepted in posterior Sanskrit literature.[6] The oldest Hindu compound name for rectilinear figures ending with *srakti* meaning the angle or corner, is *catuḥsrakti* ('quadrangle') which occurs in the *Saṃhitās* and the *Brāhmaṇas* (*c*. 2000 B.C.). In the time of the *Śrauta-sūtra* (*c*. 2000—1500 B. C.), was introduced another kind of name consisting of compounds of number names with *aśra* or *asra*, e.g. *tryasra*, *caturasra*, etc. Though these words *aśra* and *asra*, ordinarily mean "corner" or "angle", in compound names for rectilinear figures, they are

1. The subject has been treated fully in an article of Datta, B. *JASB* (new series), Vol. XXVI (1930), pp. 283-299 ; see also his *Śulba*, pp. 221-6.
2. The term *karṇa* is used to denote the hypotenuse of a right-angled triangle (*vide infra*).
3. iv. 7-8.
4. Some writers are of opinion that the word *koṇa* is derived from Greek sources, but we do not think so.
5. *Sūtra* 19, 25.
6. See for instance, *Pariśiṣṭas of the Atharva-Veda*, xxiii. 1 ; 5 ; xxv, 1, 3, 6, 7, etc. ; *Artha-śāstra* of Kauṭilya, ii. 11, 29.

sometimes found to denote "side". It is perhaps noteworthy that like the early Hindus, the early Greeks also followed the usage of naming the rectilinear figures according to the number of sides as well as of angles.[1] But while with the Hindus the angle-nomenclature is older than the side-nomenclature, with the Greeks quite the contrary is the case.[2]

Triangles are classified according to the sides : *sama-tribhuja* ('equilateral triangle'), *dvisama-tribhuja* ('isosceles triangle') and *viṣama-tribhuja* ('scalene triangle'). The classification according to the angles is not found here. Only the right-angled triangle is called by the name *jātya-tribhuja* by Brahmagupta and others.[3] The oblique triangles are grouped according as the perpendicular *(lamba)* from the vertex on the base falls inside or outside the figure, viz. *antarlamba* ('in-perpendicular') and *bahir-lamba* ('out-perpendicular'). In the *Taittirīya Saṃhitā* (c. 3000 B. C.), the *Brāhmaṇa* (c. 2000 B. C.) and the *Śulba*, an isosceles triangle is called *praüga*, derived probably from *pra* + *yuga*, meaning "the fore part of the shafts of a chariot." A rhombus is similarly called *ubhayataḥ praüga* ('*praüga* on both sides').[4]

In the *Śulba*[5] the diagonal of a rectilinear figure is called the *akṣṇa* or *akṣṇayā* ('that which goes across or transversely', i.e. 'the cross line') ; also *karṇa*, meaning 'the line going across the *karṇa* or angle.', or 'the line going across from corner to corner'. Referring to the instrument of measurement, it is sometimes termed the *akṣṇayā-veṇu* ('diagonal bamboo-rod') or *akṣṇayā-rajju* ('diagonal cord'). Out of all these only the term *karṇa* has survived, others have become obsolete.

The classification of quadrilaterals according to the sides as well as the angles is found as early as the *Sūryaprajñapti*. There are generally distinguished five kinds of quadrilaterals ; *sama-caturbhuja* ('square'), *āyata-caturbhuja* ('rectangle'), *dvisama-caturbhuja* ('isosceles trapezium'), *trisama-caturbhuja* ('equi-trilateral trapezium'), and *viṣama-caturbhuja* ('quadrilateral of unequal sides'). Similarly

1. Tropfke, J. *Geschichte der Elementar-Mathematik*, (1923), Bd. IV, pp. 60-1.
2. The conjecture of S. Gandz that "the observation of the corners and angles and the classification according to their number seem to be distinctly Greek, a specific invention of the Greek science, based upon the introduction of angle-geometry" is erroneous. Vide his article on "The origin of angle-geometry" in *Isis*, XII, pp. 452-481 ; more particularly p. 473.
3. The Sanskrit word *jātya* means "noble", "well-born", "genuine " The name *jātya-tribhuja* for the right-angled triangle seems to imply that all other triangles are derived from it.
4. Datta, *Śulba*, pp. 223f.
5. *Ibid*, pp. 224f.

we have the *sama-caturasra*, *āyata-caturasra*, *dvisama-caturasra* and *viṣama-caturasra* for those figures (*caturbhuja*=*caturasra*=quadrilateral). In the *Śulba*, the square is generally called *sama-caturasra* and the rectangle *dīrgha-caturasra* ('longish quadrilateral').

Circle

In early geometry, the circle was termed *maṇḍala* ('round') *pari-maṇḍala* ('round on all sides') ; the circumference, *pariṇāha* ('surrounding boundary line') ; the diameter, *viṣkambha* or *vyāsa* ('breadth') ; and the centre, *madhya* ('middle'). The last term had, however, wider use so as to denote the middlemost point of a square, rectangle or line. So also the terms *viṣkambha* and *vyāsa*. In Prakrita works of the fourth century before the Christian era, the term *parimaṇḍala* is used to denote the ellipse.[1] In later geometry, the term for the circle is *vṛtta*[2] and for the centre *kendra*.[3] The significance of the terms *vyāsa* and *viṣkambha* has now become fixed for the diameter of a circle. The radius is called *vyāsārdha or viṣkambhārdha* ('semidiameter'). These terms occur as early as the works of Umāsvāti (*c*. 150).[4] Still earlier in the *Āpastamba Śulba*, we find the term *ardha-vyāyāma*, having the identical significance.

Surface and Area

In the early Hindu geometry, a plane surface bounded by a figure was called by the term *kṣetra* and its area by *bhūmi*. Occasionally, however, the term *kṣetra* was employed also to signify area. In the canonical works of the Jainas (500–300 B.C.), a plane surface is termed *pratara* ('expanse'), and it is defined as that which is obtained by multiplying line by line. In posterior geometry, the *bhūmi*, together with its synonyms *bhū*, *mahī*, etc., signifying earth, denotes the ground or base of a plane figure ; the area is called *kṣetra-phala*, *kṣetra-gaṇita* or simply *phala*, *gaṇita*. These terms carry the concept of specific operations of mensuration by breaking up the figure into smaller portions and calculating them so that the area is what is obtained as the result (*phala*) of such calculation (*gaṇanā*). Another term is more explicit. It is *sama-koṣṭha-miti* ('the measure of like compartments' or 'the measure of the number of equal squares'). A curved surface or surface of a solid is called its *pṛṣṭha* ('back'), from *dharā-pṛṣṭha* (or 'the back of the earth') which is rounded. The term for the superficial area of a solid is *pṛṣṭha-phala*.

1. *Dhammasaṅgani* 617 ; *Bhagavatī-sūtra*, *Sūtra* 724-6. See Datta, *Hindu Contribution, to Mathematics* p. 8.
2. See *Bhagavatī-sūtra*, *Sūtra* 724-6.
3. In Hindu astronomy the term *kendra* is used to signify the anomaly.
4. See his *Tattvārthādhigama-sūtra-bhāṣya*, iv. 14 ; *Jambūdvīpa-samāsa*, ch. iv.

4. Typical Propositions of Early Geometry[1]

The *Śulba-sūtras*, which form a part of the Vedic literature of the Hindus, deal with the construction of firealtars for sacrificial purposes. At present we know of seven *Śulba-sūtras*, although it is quite likely that many more such works existed in ancient times. According to European scholars, these *Sūtras* were composed in the period 800 to 500 B.C., but they are probably much older. The *vedīs* ('altars') dealt with in these *sūtras* are of various forms. Their construction requires a knowledge of the properties of the square, the rectangle, the rhombus, the trapezium, the triangle and the circle. The geometrical propositions involved in the constructions are the following:

Constructions

(1) [2] To divide a line into any number of equal parts.

(2) [3] To divide a circle into any number of equal areas by drawing diameters.

(3) [4] To divide a triangle into a number of equal and similar areas.

(4) [5] To draw a straight line at right angles to a given line.

(5) [6] To draw a straight line at right angles to a given straight line from a given point on it.

(6) [7] To construct a square on a given side.

(7) [8] To construct a rectangle of given sides.

(8) [9] To construct an isosceles trapezium of given altitude, face and base.

(9) [10] To construct a parallelogram having given sides at a given inclination.

(10) [11] To construct a square equal to the sum of two different squares.

(11) [12] To construct a square equivalent to two given triangles.

---

1. For details consult Dattta, B., *The Science of the Śulba*, Calcutta, (1932).
2. The knowledge of this construction is throughout assumed. It was probably done by drawing parallels, as in Euclid. The following construction shows this surmise to be correct.
3. *BŚl*, ii. 73-4; *ĀpŚl*, vii. 13-14.
4. *BŚl*, iii. 256; See Datta, *Śulba*, p. 46.
5. *KŚl*, i.3.
6. *Ibid.*
7. *ĀpŚl*, viii. 8-10; xi. 1; i. 7; i.2; *BŚl*, i. 22-28, 29-35, 42-44; iii. 13. *TS*, v, 2.5.1.; ff. *MaiS*, iii. 2.4; *KṭS*, xx. 3.4; *KapS*, xxxii. 5.6; *ŚBr* x. 2.3.8 (2000 B.C.), etc.
8. *BŚl*, i 36-40.
9. *BŚl*, i. 41; *ĀpŚl*, v. 2-5.
10. *ĀpŚl*, xix. 5.
11. *BŚl*, i. 51-52; *ĀpŚl*, ii. 4-6; *KŚl*, ii. 22, iii. 1.
12. This follows from the above.

(12) [1] To construct a square equivalent to two given pentagons.
(13) [2] To construct a square equal to a given rectangle.
(14) [3] To construct a rectangle having a given side and equivalent to a given square.
(15) [4] To construct an isosceles trapezium having a given face and equivalent to a given square or rectangle.
(16) [5] To construct a triangle equivalent to a given square.
(17) [6] To construct a square eqivalent to a given isosceles triangle.
(18) [7] To construct a rhombus equivalent to a given square or rectangle.
(19) [8] To construct a square equivalent to a given rhombus.

Theorems

The following theorems are either expressly stated or the results are implied in the methods of construction of the altars of different shapes and sizes :

(1) The diagonals of a rectangle bisect each other. They divide the rectangle into four parts, two and two (vertically opposite) of which are equal in all respects.[9]

(2) The diagonals of a rhombus bisect each other at right angles.

(3) An isosceles triangle is divided into two equal halves by the line joining the vertex to the middle point of the base.[10]

(4) The area of a square formed by joining the middle points of the sides of a square is half that of the original one.

(5) A quadrilateral formed by the lines joining the middle points of the sides of a rectangle is a rhombus whose area is half that of the rectangle.

(6) A parallelogram and rectangle on the same base and within the same parallels have the same area.

(7) The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.

(8) If the sum of the squares on two sides of a triangle be equal to the square on the third side, then the triangle is right-angled.

1. *BŚl*, iii. 68, 288 ; *KŚl*, iv. 8.
2. *BŚl*, i. 58 ; *ĀpŚl*, ii. 7 ; *KŚl*, iii. 2, 3.
3. *ĀpŚl*, iii. 1 ; *BŚl*, i. 53.
4. *BŚl*, i. 55 ; *ŚBr*, x. 2. 1. 4.
5. *BŚl*, i.56.
6. *KŚl*, iv. 5.
7. *BŚl*, i. 57 ; *ĀpŚl*, xii. 9 ; *KŚl*, iv. 4.
8. *KŚl*, iv. 6.
9. Implied in *BŚl*, iii. 168-9, 178.
10. *BŚl*, iii. 256.

The Baudhāyana Theorem

Theorem 7 given above has been stated by Baudhāyana (*c.* 800 B.C.) in the following words :

"The diagonal of a rectangle produces both areas which its length and breadth produce separately."[1]

Āpastamba[2] and Kātyāyana[3] give the above theorem in almost identical terms. The theorem is now universally associated with the name of the Greek Pythagoras (*c.* 540 B.C.) though "no really trustworthy proof exists that it was actually discovered by him"[4]. The Chinese knew the numerical relation for the particular case $3^2+4^2=5^2$ probably in the time of Chou-Kong (d. 1105 B.C.)[5]. The *Kahun* Papyrus (*c.* 2000 B.C.) contains four similar numerical relations, all of which can be derived from the above one.[6] As for the Hindus, one instance of that kind, $39^2=36^2+15^2$, occurs in the *Taittirīya Saṃhitā*[7] (before 2000 B.C.). It should be noted that this instance is different from that known to other early nations.

Although particular instances of the theorem are found amongst several ancient nations, the first enunciation of the theorem in its general form is found in India. It cannot be said what made Baudhāyana give the theorem in the general form. It is not improbable that he possessed a proof of the theorem. But what this proof was will never be known with certainty. Bürk, Hankel, Thibaut and Datta are of opinion that Baudhāyana knew a proof of the theorem.[8] It is conjectured that this proof may have been one of the following :

Hindu Proofs

(i) Let *ABCD* be a given square. Draw the diagonal *AC*; produce *AB* and cut off *AE* equal to *AC* (Fig. 2). Construct the square *AEFG* on *AE*. Join *DE* and on it construct the square *DHME*. Complete the construc-

---

1. *BŚl*, i. 48 :—"*Dīrghacaturasrasyākṣṇayārajjuḥ pārśvamānī tiryaṅmānī ca yatpṛthagbhūte kurutastadubhayaṁ karoti*".
2. *ĀpŚl*, i. 4.
3. *KŚl*, ii. 11.
4. Heath, *Greek Math.*, Vol. I, p. 144f.
5. Mikami, Y., *The Development of Mathematics in China and Japan*, Leipzig (1913), p.7.
6. These are $1^2 + (\frac{3}{4})^2 = (1\frac{1}{4})^2$,
$2^2 + (1\frac{1}{2})^2 = (2\frac{1}{2})^2$,
$8^2 + 6^2 = 10^2$,
$16^2 + 12^2 = 20^2$.
7. vi. 2. 4. 6.; It also occurs in the *Śatapatha Brāhmaṇa*, x. 2. 3. 4.
8. Datta, *The Science of the Śulba*, ch. ix.

tion as indicated in Fig. 2. Now the square *DHME* is seen to be comprised of four right-angled triangles each equal to *DAE* and the small square *ANPQ*. This square will be easily recognised to be equal to the square *CRFS* and triangles equal to the rectangles *AERD* and *ABSG*. Therefore, the square *DHME* is equal to the sum of t˙e squares *ABCD* and *AEFG*. Hence the theorem.

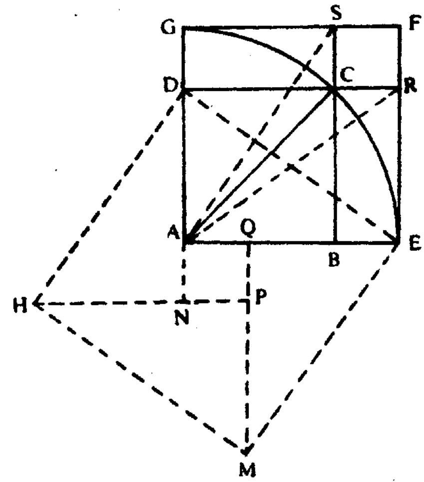


FIG. 2

It might be mentioned that constructions like the above are necessary in the usual course in the *Śulba*.

(ii) Let *ABC* be a right-angled triangle of which the angle *C* is a right-angle. From *C* draw the perpendicular *CD* on *AB*. Then the triangle *ABC*, *ACD* and *CBD* are similar.

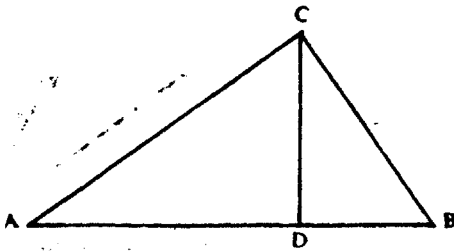


FIG. 3

Therefore, $AB : AC :: AC : AD$,

Or, $AC^2 = AB.AD$

Similarly, $CB^2 = AB.DB$

Adding we get $AC^2 + CB^2 = AB^2$

This proof is given by *Bhāskarācārya*,[1] and does not occur in the west until 1693 when it was rediscovered in Europe by Wallis.

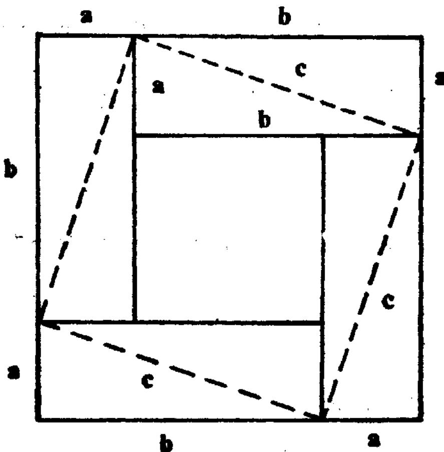


FIG. 4(a)

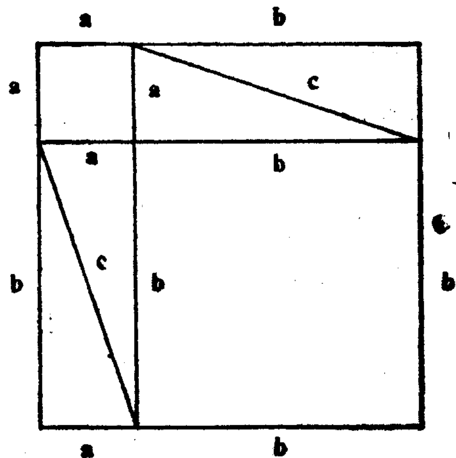


FIG. 4(b)

1. *Cf.* Colebrooke, ***Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhaskara, London, 1817***, pp. 221.2

(iii) Let a, b, c be the sides of a right-angled triangle. Taking four such triangles they are arranged as in Fig. 4 (a), inside a square whose side is equal to the hypotenuse of the given triangle.

Obviously then,

$$c^2 = 4\left(\frac{ab}{2}\right)+(b-a)^2 = a^2 + b^2.$$

This proof was anticipated by the Chinese by several centuries.[1]

The technique employed in this proof was used by Āpastamba for the enlargement of a square. Thus to construct a square whose side will exceed a side b of a given square by a, add, says Āpastamba, on the two sides of the given square two rectangles whose lengths are equal to b and breadths to a; then add on the corner a square whose sides are equal to the increment a. Thus will be obtained a square with a side equal to $a+b$ (Fig. 4b). A similar method was taught by Baudhāyana.[2]

Particular Case :

The particular case of the above theorem relating to the diagonal of a square has been stated thus :

"The diagonal of a square produces an area twice as much,"

The statement is given in all the *Śulba-Sūtras*[3] and the theorem has been used for "doubling the square" at several places. Instances of its use are found in the *Taittirīya* (before 2000 B. C.) and other *Saṃhitās*, and can be traced back to the *Ṛgveda* (before 3000 B. C.)

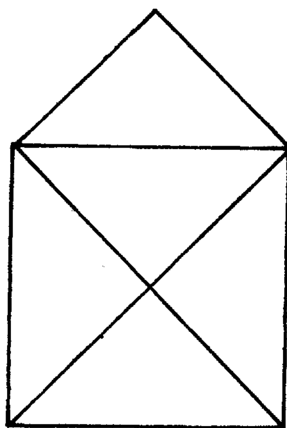


FIG. 5

Thibaut says : "The authors of the *sūtras* do not give us any hint as to the way in which they found their proposition regarding the diagonal of a square; but we suppose that they, too, were observant of the fact that the square of the diagonal is divided by its diagonals into four triangles, one of which is equal to half the first square (Fig. 5). This is at the same time an immediately convincing proof of the Pythagorean proposition as far as squares or equilateral rectangular triangles are concerned".[4]

1. Mikami, *l. c.*, p. 5.
2. See Datta, *The Science of the Śulba*, p. 117.
3. *BŚI*, i. 45; *ĀpŚI*, i. 5; *KŚI*, ii. 12; etc.
4. Thibaut, *Śulbasūtras*, p. 8.

5. Measurement of Triangles

Area of a triangle

The method for finding the area of a triangle that was known in the *Śulba*[1] was

$$\text{Area} = \tfrac{1}{2}\ (\text{base} \times \text{altitude})$$

And that was one of the methods followed in later times. Āryabhaṭa I says :

"The area of a triangle is the product of the perpendicular and half the base"[2].

According to Brahmagupta :

"The product of half the sums of the sides and countersides of a triangle or a quadrilateral is the rough value of its area. Half the sum of the sides is severally lessened by the three or four sides, the square-root of the product of the remainders is the exact area".[3]

That is to say, if a, b, c, d, be the four sides of a quadrilateral taken in order, we have

$$\text{Area} = \frac{c+d}{2} \times \frac{a+b}{2}, \text{ roughly}$$

$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ exactly,}$$

where $$s = \tfrac{1}{2}(a+b+c+d).$$

In case of a triangle $d=0$; so that we get

$$\Delta = \frac{c}{2} \times \frac{a+b}{2}, \text{ roughly}$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \text{ exactly.}$$

The second formula was given before by the Greek Heron of Alexandria (*c.* 200).[4] Pṛthūdakasvāmi calculates by these methods the area of the triangle (14, 15, 13) to be 98 roughly, 84 exactly.

Śrīdhara says that the exact value of a triangle will be given by the formulae[5]

$$\Delta = \tfrac{1}{2}\ (\text{base} \times \text{altitude}),$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

1. See Datta, *The Science of the Śulba*, p. 96.
2. *Ā*, i. 6.
3. *BrSpSi*, xii. 21.
4. Heath, *History of Greek Mathematics*, II, p. 321.
5. *Triś*, R. 43.

Mahāvīra[1], Āryabhaṭa II[2], and Śrīpati[3] teach both these accurate methods as well as the rough one of Brahmagupta. Bhāskara II[4] adopts the formula

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}.$$

Segments and Altitudes

Bhāskara I (629) writes :

"In a triangle the difference of the squares of the two sides or the product of their sum and difference is equal to the product of the sum and difference of the segments of the base. So divide it by the base or the sum of the segments ; add and subtract the quotient to and from the base and then halve, according to the rule of concurrence. Thus will be obtained the values of the two segments. From the segments of the base of a scalene triangle, can be found its altitude."[5]

That is to say

$$a^2-b^2=(a+b)(a-b)=c_1^2-c_2^2=(c_1+c_2)(c_1-c_2),$$

also

$$c_1+c_2=c.$$

Therefore

$$c_1-c_2=\frac{a^2-b^2}{c},$$

Hence

$$c_1=\tfrac{1}{2}\left(c+\frac{a^2-b^2}{c}\right),$$

$$c_2=\tfrac{1}{2}\left(c-\frac{a^2-b^2}{c}\right),$$

$$h=\sqrt{a^2-c_1^2}=\sqrt{b^2-c_2^2}.$$

By means of these formulae Bhāskara I finds the segments (9, 5 ; 35, 16) of the bases (14, 51), altitudes (12, 12) and areas (84, 306) of the scalene triangles (13, 15, 14) and (20, 37, 51).

Brahmagupta (628) gives the same set of formulae. He says :

"The difference of the squares of the two sides being divided by the base, the quotient is added to and subtracted from the base ; the results, divided by two, are the segments of the base. The square-root of the square of a side as diminished by the square of the corresponding segment is the altitude."[6]

1. *GSS*, vii. 7, 50.
2. *MSi*, xv. 66, 69, 78.
3. *Si Śe*, xiii. 30.
4. *L*, p. 41.
5. Vide his commentary on *Ā*, ii. 6.
6. *BrSpSi*, xii. 22.

Pṛthūdakasvāmi proves these formulae in the same way as Bhāskara I and also applies them to the latter's first example (13, 15, 14).

Śrīdhara first finds the area of the triangle by means of the formula $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$ and then deduces the segments and perpendicular. His rules are :

"Twice the area of the triangle divided by the base is the altitude. (Then there will be two right-angled triangles of which) the uprights are equal to that altitude, bases are the segments and hypotenuses, the two sides (of the given triangle)."[1]

Mahāvīra says :

"Divide the difference between the squares of the two sides by the base. From this quotient and the base, by the rule of concurrence, will be obtained the values of the two segments (of the base) of the triangle ; the square-root of the difference of the squares of a segment and its corresponding side is the altitude : so say the learned teachers."[2]

Āryabhaṭa II writes :

"In a triangle, divide the product of the sum and difference of the two sides by the base. Add and subtract **the** quotient to and from the base and then halve. The results will be the segments corresponding to the greater and smaller sides respectively. The segment corresponding to the smaller side should be considered negative, if it lie outside the figure. The square-root of the difference of the squares of a segment and its corresponding side is the perpendicular."[3]

Similar rules are given by Śrīpati[4] and Bhāskara II[5]. The latter gives in illustration a case of a scalene triangle whose base is 9, and sides 10, and 17. There the segments are 6 and 15, and perpendicular 8.

Circumscribed Circle

Brahmagupta says :

"The product of the two sides of a triangle divided by twice the altitude is the heart-line (*hṛdaya-rajju*). Twice it is the diameter of the circle passing through the corners of the triangle and quadrilateral."[6]

Pṛthūdakasvāmi proves it substantially as follows :

1. *Triś*, R. 50.
2. *GSS*, vii. 49.
3. *MSi*, xv. 76-7.
4. *SiŚe*, xii. 29.
5. *L*, p. 40.
6. *BrSpSi*, xii. 27.

Let ABC be a scalene triangle. Draw AD perpendicular to BC. Produce it to A' making $A'D = AD$. Let O be the centre of the circle circumscribing the triangle ABC. Join OA, OC. Triangles BAA' and OAC are similar.

Therefore, $AB : OA :: AA' : AC$.

Hence, $$OA = \frac{AB. AC}{AA'}$$

Or, $$R = \frac{cb}{2h},$$

where R denotes the radius of the circumscribed circle.

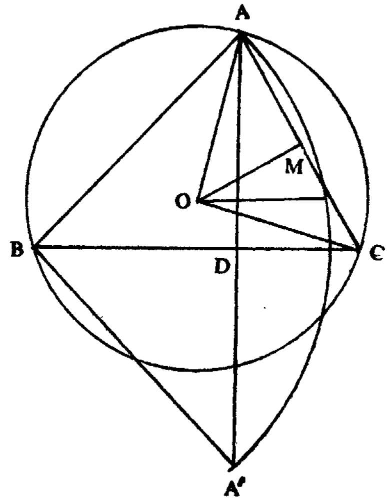


Fig. 6.

Mahāvīra writes :

"In a triangle, the product of the two sides divided by the altitude is the diameter of the circumscribed circle."[1]

Example[2] : The circum-diameter of the triangle (14, 13, 15) is 16¼.

Śrīpati states :

"Half the product of the two sides divided by the altitude is the heart-line."[3]

Inscribed Circle

To find the radius of a circle inscribed in a triangle (or quadrilateral, when possible) whose area as well as perimeter are known, Mahāvīra gives the following rule ;

"Divide the precise area of a figure other than a rectangle by one-fourth of its perimeter ; the quotient is stated to be the diameter of the inscribed circle."[4]

That is to say, if r denote the radius of the circle inscribed within the triangle (a, b, c), we shall have

$$r = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)},$$

where

$$2s = a + b + c.$$

1. *GSS*, vii. 213½.
2. *GSS*, vii. 219½.
3. *SiSe'*, xiii. 31.
4. *GSS*, vii. 223½.

Similar triangles

The properties of similar triangles and parallel lines were known to the ancient Hindus.[1] For example, take the case of the Mount Meru or Mandara. It has been described in the early canonical works of the Jainas as follows :

"At the centre of Jambūdvipa, there is known to be a mountain, Mandara by name, whose height above (the earth) is 99000 *yojanas*, whose depth below is 1000 *yojanas*, its diameter at the base is $10090\frac{10}{11}$ *yojanas*, at the ground 10000 *yojanas*. Then (its diameter) diminishes by degrees until at the top it is 1000 *yojanas*. Its circumference at the base is $31910\frac{3}{11}$ *yojanas*, at the ground 31623 *yojanas*, and at the top a little over 3162 *yojanas*. It is broader at the base, contracted at the middle and (still) shorter at the top and is of the form of a cow's tail (i. e. a truncated right cone)."[2]

To find the diameter of any other section parallel to the base, Jinabhadra Gaṇi (*c*. 560) gives the following rule :

"Wherever is wanted the diameter (of the Mandara) : the descent from the top of the Mandara divided by eleven and then added to a thousand will give the diameter. The ascent from the bottom should be similarly (divided by eleven) and the quotient subtracted from the diameter of the base : what remains will be the diameter there (i.e. at that height) of that (Mandara)."[3]

It is stated further :

"Half the difference of the diameters at the top and the base should be divided by the height ; that (will give) the rate of increase or decrease on one side ; that multiplied by two will be the rate of increase or decrease on both sides ; in going from either end of the mountain,

"Subtract from the diameter of the base of the mountain the diameter at any desired place : what remains when multiplied by the denominator (meaning eleven) will be the height (of that place)."[4]

1. See Datta, Bibhutibhusan "Geometry in the Jaina Cosmography", *Quellen und Studien z. Gesch. d. Math.*, Ab. B, Bd. 1., 1930, pp. 249ff.
2. *Jambūdvipa-prajñapti*, *Sūtra* 103.
3. *Vṛhat Kṣetra-samāsa*, i. 307-8.
4. *Ibid*, i. 309-11.

All these rules will follow at once from the following general formulae ;

$$a = \frac{D-d}{2h}x,$$

$$\delta = a + \frac{D-d}{h}x,$$

$$y = (D-\delta')\frac{h}{D-d},$$

$$b = \frac{D-d}{2h}y,$$

$$\delta' = D - \frac{D-d}{h}y.$$

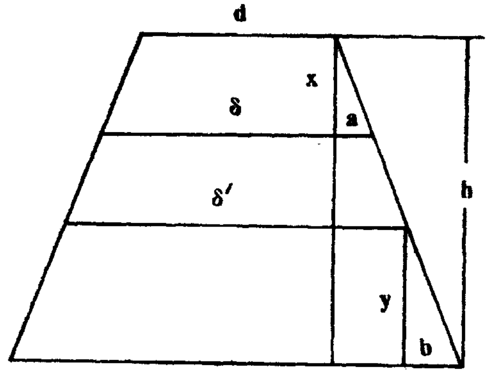


FIG. 7

Rules similar to those stated above and hence the general properties leading to them, were known to the people long before Jinabhadra Gaṇi. For as early as the second century before the Christian era (or after) Umāsvāti correctly observed that in case of the Mount Meru, "for every ascent of 11000 *yojanas*, the diameter diminishes by 1000 *yojanas*."[1]

Again, "Half the difference between the breadths at the source and the mouth being divided by 45000 *yojanas*, and the quotient multiplied by two will give the rate of increase (of the breadth) on both sides, in case of rivers."[2] (45000 *yojanas* is the length of a river).

They are found even in the early canonical works (500-300 B. C.). According to the Jaina cosmography, the Salt Ocean is annular in shape, having a breadth of 200000 *yojanas*. In the undisturbed state its height as well as depth are said to be varying continuously from its either banks till at distances of 95000 *yojanas* from the banks where the height is 16000 *yojanas* and the depth 1000 *yojanas*. The radial section of the Salt Ocean in the calm state will be represented by Fig 8,

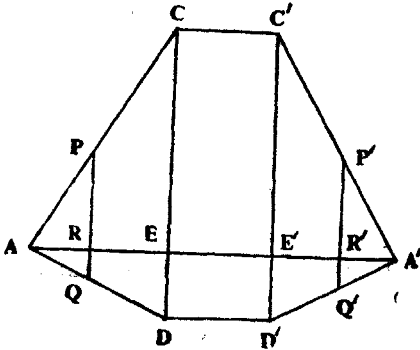


FIG. 8

where $AE = A'E' = 95000$ *yojanas*,
$CE = C'E' = 16000$ *yojanas*,
$ED = E'D' = 1000$ *yojanas*,
and $EE' = 10000$ *yojanas*.

1. *Tattvārthādhigama-sūtra-bhāṣya*, iii. 9.
2. *Jambūdvīpa-samāsa*, ch, iv.

It is described in the *Jivābhigama-sūtra* that "from either bank of the Salt Ocean, for proceeding every 95 *padas*, the height is known to be increased by 16 *padas* and so on, until on proceeding to 95000 *yojanas*, the height is known to be increased to 16000 *yojanas* "[1]
These can be easily verified thus :

From the properties of similar triangles

$$QR = \frac{ED \times AR}{AE} = \frac{1}{95} AR,$$

$$PR = \frac{EC \times AR}{AE} = \frac{16}{95} AR.$$

If $AR = 95x$, where x is any unit of measurement, then $QR = x$, $PR = 16x$.

Again it is stated in the *Jambūdvīpa-prajñapti*[2] that at a height of 500 *yojanas* above the ground the breadth of the Mount Mandara is $9954\frac{6}{11}$ *yojanas*, while at 63000 *yojanas* above it is $4272\frac{8}{11}$ *yojanas*. These values, as can be easily verified, tally with the general formulae (p. 21).

6. Measurement of Quadrilaterals

Area

It should be noted at the outset that four sides alone are not sufficient to determine the true shape of a quadrilateral and consequently its size. For, there can be formed various quadrilaterals with the same four sides. Hence in order to make a quadrilateral determinate we must know, besides the sides, another element such as a diagonal, the altitude of a corner, or an angle. Thus Āryabhaṭa II remarks :

"The mathematician who wishes to tell of the area or the altitudes of a quadrilateral without knowing a diagonal, must be a fool or a blunderer."[3]

Bhāskara II writes :

"The diagonals of a quadrilateral (whose four sides are given) are uncertain. How can, then, the area be determinate ? The diagonals as calculated by previous teachers will be true only in case of quadrilaterals (of a particular kind) contemplated by them, but not in case of others. For with the same (four) sides, there can be various other pairs of diagonals and consequently the area also is manifold. In a quadrilateral, when two opposite corners are so drawn as to bring the sides contiguous to them inwards, the diagonal

1. *Jivābhigama-sūtra*, *sūtra* 172.
2. *Sūtra* 104-5.
3. *MSi*, xii. 70.

joining them is shortened, while the other two corners bulge outwards and consequently their diagonal is lengthened. So it has been stated (just before) that with the same sides there can be other pairs of diagonals. Without specifying one of the altitudes or diagonals, how can one ask to find the other of them and also the area, as these are truly indeterminate? The questioner who does not know the indeterminate nature of a quadrilateral must be a blunderer; still more so is he, who answers such a problem."[1]

Brahmagupta's Formula

To find the area (A) of an inscribed convex quadrilateral whose sides are a, b, c, d, Brahmagupta (628) gives the following formula :[2]

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $$2s = a + b + c + d.$$

This formula has been reproduced by Śrīdhara[3] (900), Mahāvīra[4] (850) and Śrīpati[5] (1039). None of these writers has expressly mentioned the limitation that it holds only for an inscribed figure. Still it seems to have been implied by them. So this appears from the particular remark of Bhāskara II that the formula holds only in case of a special kind of quadrilaterals contemplated by them. Further we find that the examples of quadrilaterals, viz. (4, 13, 14, 13), (25, 25, 39, 25) and (25, 39, 60, 52) given by Śrīdhara[6] and Pṛthūdakasvāmi[7] and those, namely (14, 36, 61, 36), (169, 169, 407, 169) and (125, 195, 300, 260) given by Mahāvīra[8], in illustration of the above formula, are all of the cyclic variety. Bhāskara II has shown that in the other cases, the above formula gives only an approximate value of the area of a quadrilateral.[9]

Diagonal, altitude and segment

Āryabhaṭa I says:

"The two sides (severally) multiplied by the altitude and divided by their sum will give the perpendiculars let fall on them from the point of

1. *L*, p. 44.
2. *BrSpSi*, xii. 21.
3. *Triś*, R. 43.
4. *GSS*, vii. 50.
5. *SiŚe*, xiii, 28.
6. *Triś*, Ex. 78, 79, 80.
7. Vide his commentary on *BrSpSi*, xii. 21. Elsewhere (xii. 26) he finds the circum-radii of these quadrilaterals.
8. *GSS*, vii, 57, 58, 59. Compare also vii. 215½, 216½, 217½ where it is required to find the diameters of the circles circumscribing these very quadrilaterals.
9. *L*, p. 41.

intersection of the diagonals. Half the sum of the two sides multiplied by the altitude should be known as the area."[1]

$$h_1 = \frac{ah}{a+c}$$

$$h_2 = \frac{ch}{a+c}$$

$$\text{Area} = \tfrac{1}{2}h\,(a+c).$$

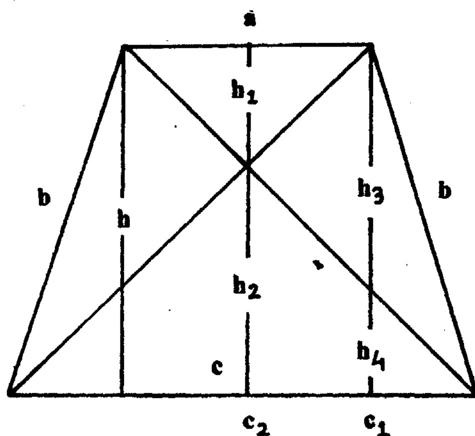


FIG. 9.

Brahmagupta writes :

"In an isosceles trapezium[2] the square-root of the sum of the products of the sides and countersides is the diagonal. The square-root of the square of the diagonal as diminished by the square of half the sum of the face and base, is the altitude."[3]

$$d = \sqrt{ac+b^2}, \qquad h = \sqrt{d^2 - \left(\frac{a+c}{2}\right)^2}.$$

"The upper and lower portions of the diagonal or the altitude at the junction of the two diagonals or of a diagonal and an altitude, will be given by the corresponding segments of the base divided by their sum and multiplied again by the diagonal or altitude, as the case may be."[4]

$$h_3 = \frac{c_2 h}{c_1+c_2}, \qquad d_1 = \frac{c_2 d}{c_1+c_2},$$

$$h_4 = \frac{c_1 h}{c_1+c_2}, \qquad d_2 = \frac{c_1 d}{c_1+c_2}.$$

For quadrilaterals other than isosceles trapeziums, Brahmagupta gives the following rules :

1. *Ā*, ii. 8.
2. The Sanskrit term is *avisama-caturasra,* meaning literally "the quadrilateral not of unequal sides." Brahmagupta classifies quadrilaterals (*caturasra, caturbhuja*) into five varieties : *sama-caturasra* (square), *āyata-caturasra* (rectangle), *dvisama caturasra* (isosceles trapezium), *trisama-caturasra* (trapezium with three equal sides) and *visama-caturasra* (quadrilateral of unequal sides). Hence *aviṣama-caturasra* must mean all except those of the last class. But here more particularly the isosceles trapezium is meant.
3. *BrSpSi,* xii. 23.
4. *BrSpSi,* xii. 25.

"Considering two scalene triangles within the quadrilaterals[1] by means of the two diagonals, find separately the segments of the base in them by the method taught before ; and thence the two altitudes."[2]

"Supposing two scalene triangles within the quadrilateral, with the diagonals as bases, find in each of them separately the segments of the base. They will be the portions of the diagonals above and below their point of intersection. The lower portions of the diagonals are taken to be the sides of another triangle whose base is the same as that of the given quadrilateral. Its altitude is the lower portion of the perpendicular (to the base through the junction of the diagonals). The upper portions of it will be obtained by subtracting this portion from half the sum of the two altitudes."[3]

"At the intersection of the diagonals and perpendiculars, the lower segment of a diagonal and of a perpendicular can be found by proportion. On subtracting these segments from the whole, the upper portions will be found. Such is (the method) also in the needle (i.e. the intersection of two opposite sides produced) and the intersection (of a prolonged side and perpendicular)."[4]

Śrīdhara states :

"To find the altitude of a trapezium,[5] suppose a triangle whose base is the difference of the base and face of the trapezium and whose sides are the same as those at the flanks of the given figure ; (and then proceed as in the case of finding the altitude of a triangle)."[6]

Mahāvīra's rule will be clear from the following problem with reference to which it has been defined :

AB, *CD* are two vertical pillars. *AE*, *CF* are two strings joining the tops *A* and *C* of the these pillars to points *E* and *F* on the ground. *PQ* is the perpendicular from the point of intersection of the strings. It has been named "the inner perpendicular."

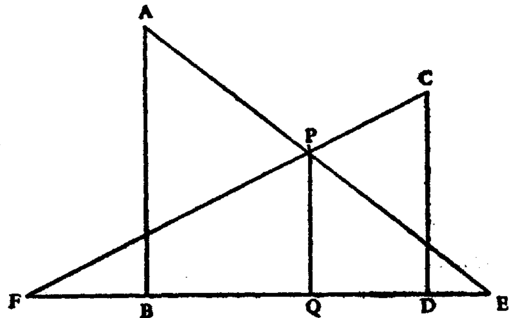


FIG. 10.

Mahāvīra says :

"Divide each pillar by its distance from (the farthest point of contact of) the string (with the ground), divide again the quotients by their

1. The Sanskrit term is *viṣama-caturasra*. As pointed out just before, it denotes "a quadrilateral of unequal sides" including a trapezium.
2. *BrSpSi*, xii. 29.
3. *BrSpSi*, xii. 30-31.
4. *BrSpSi*, xii. 32.
5. The Sanskrit term is *ṛjuvadana-caturbhuja* or "the quadrilateral with parallel face."
6. *Triś*, R. 49.

sum and then multiply by the (whole) base. The results are the segments (of the base by the inner perpendicular). These being multiplied by the (first) quotients in the inverse order give the inner perpendicular."[1]

That is to say, we have

$$QF = \frac{\frac{AB}{BE}.FE}{\frac{AB}{BE}+\frac{CD}{DF}} = \frac{AB.\ DF.\ FE}{AB.\ DF+CD.\ BE},$$

$$QE = \frac{\frac{CD}{DF}.FE}{\frac{AB}{BE}-\frac{CD}{DF}} = \frac{CD.\ BE.\ FE}{AB.\ DF-CD.\ BE},$$

$$PQ = \frac{AB}{BE}\ .\ QE = \frac{CD}{DF}\ .\ QF\ .$$

Example from Mahāvīra : Find the inner perpendicular and the segments of the base caused by it in the quadrilateral (7, 15, 21, 3).

Śrīpati says :

"In an isosceles trapezium, the square-root of the sum of the products of opposite sides is the diagonal. Next I shall speak of quadrilaterals of unequal sides."[3]

Bhāskara II gives several rules. Of them we note the following :

"In a quadrilateral, assume the value of one diagonal. Then in the two triangles lying on either sides of this diagonal, it will be the base and others (i.e. the given sides of the quadrilateral) sides. Now find the perpendiculars and segments (in these triangles). Then the square of the difference of the two segments lying on the same side (i. e. taken from the same corner) being added to the square of the sum of the perpendiculars, the square-root of the resulting sum will be the second diagonal in all quadrilaterals.[4]

Gaṇeśa has demonstrated the rule substantially as follows :

Let $ABCD$ be a quadrilateral whose diagonal AC as well as the sides are known. Draw BN, DM perpendiculars to AC. Produce BN and draw DP perpendicular to it. Join DB. Then

$$DB^2 = BP^2 + DP^2,$$
$$= (BN+DM)^2 + (AN-AM)^2.$$

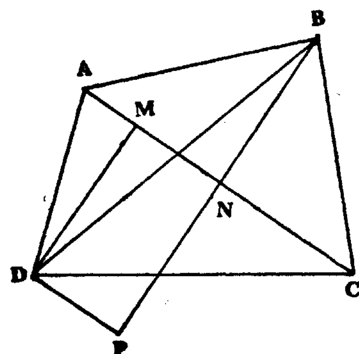
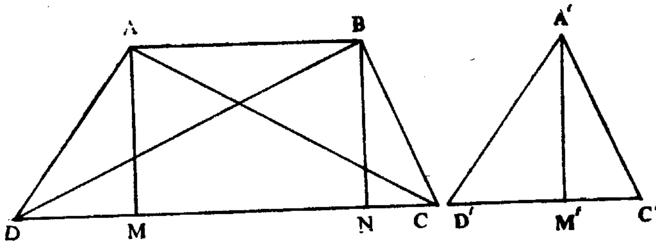


FIG. 11

1. *GSS*, vii. 180½.
2. *GSS*, vii. 187½.
3. *SiŚe*, xiii. 33,
4. *L*, p. 47f.

"Suppose a triangle whose base is equal to the difference of the face and base of a trapezium, and whose sides are the flank sides of the latter; then as in case of a triangle, find its altitude and segments of the base. Subtract from the base of the given trapezium one of the segments. The square of the remainder being added to the square of the perpendicular, the square-root of the sum is the diagonal. In a trapezium, the sum of the base and smaller flank side is greater than the sum of the face and the other flank"[1].

Gaṇeśa's Proof : Let $ABCD$ be a trapezium. Draw the perpendiculars AM, BN. Combine the two triangles ADM and BCN into one triangle $A'C'D'$. Then the altitude $A'M'$ of the new triangle is equal to the altitude of the trapezium.



FIG, 12.

Join AC and BD. Then

$$AC^2 = AM^2 + MC^2 = A'M'^2 + (DC - D'M')^2$$
$$BD^2 = BN^2 + DN^2 = A'M'^2 + (DC - C'M')^2$$

Again $A'D' - A'C' < D'C' = DC - AB$.
Thereforre $DC + A'C' > AD + AB$.

Circumscribed Circle

To find the radius of the circle described round a quadrilateral, Brahmagupta gives the following rule :

"The diagonal of an isosceles trapezium being multiplied by its flank side and divided by twice its altitude gives its heart line : in case of a quadrilateral of unequal sides, it is half the square-root of the sum of the squares of the opposite sides."[2]

Now it has been given by Brahmagupta that

$$h^2 = d^2 - \left(\frac{a+c}{2}\right)^2$$

Substituting the value of $d^2 = ac + b^2$, we get

$$h = \sqrt{(s-a)(s-c)}.$$

1, *L.* p. 48 f,

2. *BrSpSi, xii.* 26,

Hence according to the above, the radius of the circle described round the isosceles trapezium (a, b, c, b) is

$$\tfrac{1}{2} b \sqrt{\frac{ac+b^2}{(s-a)(s-c)}} .$$

In case of a quadrilateral of unequal sides the circum-radius is

$$=\tfrac{1}{2}\sqrt{a^2+c^2}=\tfrac{1}{2}\sqrt{b^2+d^2}.$$

This formula holds only in that kind of inscribed convex quadrilaterals in which the diagonals are at right angles.

Mahāvīra says :

"In a quadrilateral, the diagonal divided by the perpendicular and multiplied by the flank side, gives the diameter of the circumscribed circle."[1]

Śrīpati states all the above formulae. He says :

"In a quadrilateral, half the product of a diagonal and flank side divided by the altitude, gives the radius of the circumscribed circle. In a quadrilateral of unequal sides, half the square-root of the sum of the squares of the opposite sides is stated to be the radius and twice it the diameter of the circumscribed circle.[2]

Inscribed Circle.

We have already cited Mahāvīra's formula for the diameter of the inscribed circle.

$$\text{Diameter}=\text{Area} \div \frac{\text{Perimeter}}{4} .$$

Theorems for diagonals. Brahmagupta (628) gives two remarkable theorems for the lengths of the diagonals of an inscribed convex quadrilateral. He says :

"Divide mutually the sums of the products of the sides attached to both the diagonals and then multiply the quotients by the sum of the products of the opposite sides : the square-roots of the results are the diagonals of the quadrilateral"[3].

$$m=\sqrt{\frac{(ab+cd)\,(ac+bd)}{(ad+bc)}}$$

$$n=\sqrt{\frac{(ad+bc)\,(ac+bd)}{(ab+cd)}} .$$

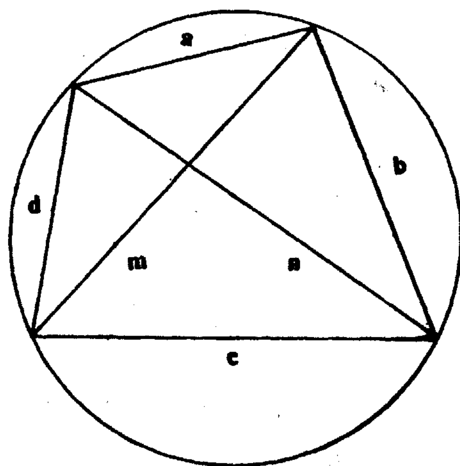


FIG. 13

1. *GSS*, vii. 213½.
2. *SiŚe*, xiii. 31-2.
3. *BrSpSi*, xii. 28.

Mahāvīra (850) writes :

"The two flank sides multiplied by the base are added (respectively) to those sides (taken reversely) multiplied by the face. Make the sums (thus obtained respectively) the multiplier and divisor, again the divisor and multiplier of the sum of the products of the opposite sides. The square-roots of the results are the diagonals."[1]

Śrīpati's (1039) enunciation[2] of the theorems is nearly the same as that of Brahmagupta.

It will be noticed that neither Brahmagupta, nor any of the posterior writers mentioned above, has expressly stated the limitation that the theorems hold only in case of inscribed convex quadrilaterals. Did they at all know it will be the question that will be naturally asked. Looking at the context, we think, it will have to be answered in the affirmative. For in the two rules just preceding the one in question, Brahmagupta teaches how to find the radii of the circles circumscribed about a quadrilateral and a triangle respectively. So in the present rule too he has in view a quadrilateral of the type which can be circumscribed by a circle. Illustrative examples given by the commentator Pṛthūdakasvāmi, as also by Mahāvīra, are all of quadrilaterals of that kind. Further Bhāskara II observed in connection with these theorems that they hold in case of quadrilaterals contemplated to be of a particular kind by their author.

7. Squaring the Circle

Origin of the Problem

The problem of 'squaring the circle,' or what was more fundamental in India, the problem of 'circling the square,' originated and acquired special importance in connexion with the Vedic sacrifices, before the earliest hymns of the *Ṛgveda* were composed (before 3000 B. C). The three primarily essential sacrificial altars of the Vedic Hindus, namely the *Gārhapatya*, *Āhavanīva* and *Dakṣiṇa*, were constructed so as to be of the same area, but of different shapes, square, circular and semi-circular. Again in constructing the fire-altars called the *Rathacakra-citi*, *Samuhya-citi* and *Paricāyya-citi*, which are mentioned in the *Taittirīya Saṃhitā* (*c.* 3000 B.C) and other works, one had to draw in each case at first a square equal in area to that of the *Śyena-citi*, viz. 7½ square *puruṣas*, and then to transform it into a circle. We find also other instances in the early Hindu works requiring the solution of the problem of circling the square and its converse.[3]

1. *GSS*, vii. 54. 2. *SiŚe*, xiii. 34.
3. See Datta, Bibhutibhusan, *Śulba*, ch. xi, for further informations on the problem.

Circling the Square

Baudhāyana writes :

"If you wish to circle a square, draw half its diagonal about the centre towards the east-west line ; then describe a circle together with the third part of that which lies outside (the square)."[1]

The same method is taught in different words also by Āpastamba[2] and Kātyāyana.[3]

Let $ABCD$ be the square which is to be transformed into a circle. Let O be the central point of the square. Join OA. With centre O and radius OA, describe a circle intersecting the east-west line EW at E. Divide EM at P, such that $EP=2\ PM$. Then with centre O and radius OP describe a circle. This circle is roughly equal in area to the square $ABCD$.

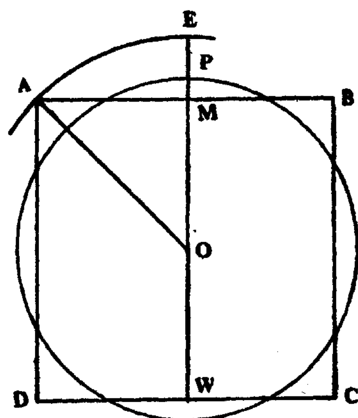


FIG. 14

Let $2a$ denote a side of the given square and r the radius of the circle equivalent to it. Then

$$OA=a\sqrt{2},\ ME=(\sqrt{2}-1)\,a.$$

Hence

$$r=a+\frac{a}{3}(\sqrt{2}-1).$$

$$=\frac{a}{3}(2+\sqrt{2}).$$

Āpastamba observes that the circle thus constructed will be inexact (*anitya*). Now, according to the *Śulba*,

$$\sqrt{2}=1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}$$

Therefore

$$r=a\times 1.1380718......$$

Squaring the Circle

Baudhāyana says :

"If you wish to square a circle, divide its diameter into eight parts ; then divide one part into twenty-nine parts and leave out twenty-eight of these, and also the sixth part (of the preceding sub-division) less the eighth part (of the last)".[4]

1. *BŚl*, i. 58,
2. *ĀpŚl*, iii. 2.
3. *KŚl*, iii. 13
4. *BŚl*, i. 59.

That is to say, if $2a$ be the side of a square equivalent to a circle of diameter d, then

$$2a=\frac{7d}{8}+\left\{\frac{d}{8}-\left(\frac{28d}{8.29}+\frac{d}{8.29.6}-\frac{d}{8.29.6.8.}\right)\right\}$$

or putting $d=2r$,

$$a=r-\frac{r}{8}+\frac{r}{8.29}-\frac{r}{8.29.6}+\frac{r}{8.29.6.8}.$$

Baudhāyana further teaches a still rough method of squaring the circle :

"Or else divide (the diameter) into fifteen parts and remove two (of them). This is the gross (value of the) side of the (equivalent) square".[1]

This method is described also by Āpastamba[2] and Kātyāyana.[3] According to it

$$a=r-\frac{2r}{15}.$$

According to Manu a square of two by two cubits is equivalent to a circle of radius 1 cubit and 3 *aṅgulis*.[4]

Dvārakānatha's Corrections : Dvārakānātha Yajvā, a commentator of the *Baudhāyana Śulba*, proposed a correction to the above formula for the transformation of a square into a circle. According to him

$$r=\left\{a+\frac{a}{3}(\sqrt{2}-1)\right\}\times\left\{1-\frac{1}{118}\right\}$$

or

$$r=a\times 1.1284272......$$

Similarly he improves the formula for the reverse operation :

$$a=r\left(1-\frac{1}{8}+\frac{1}{8.29}+\frac{1}{8.29.6}-\frac{1}{8.29.6.8}\right)\left(1+\frac{1}{2}\cdot\frac{3}{133}\right)$$

Later Formulae

In the Jaina cosmography, the earth is supposed to be a flat plane divided into successive regions of land and water by a system of concentric circles. The innermost region is one of land and is called Jambūdvīpa. It is a circle of diameter 100000 *yojanas*. Its circumference is given as a little over 316227 *yojanas* 3 *gavyūtis* 128 *dhanus* 13½ *aṅgulas* and its area as 7905694150 *yojanas* 1 *gavyūti* 1515 *dhanus* 60 *aṅgulas*.[5] It will be seen that in

1. *BŚl*, i. 60.
2. *ĀpŚl*, iii. 3.
3. *KŚl*, iii. 14.
4. *MāŚl*, i. 27.
5. See *Jambūdvīpa-prajñapti*, *Sūtra* 3 ; *Jīvābhigama-sūtra*, *Sutra* 82, 124 ; *Anuyogadvāra-sūtra*, *Sūtra* 146. Compare also *Sūryaprajñapti*, *Sūtra* 20.

calculating these values of the circumference and area from the assumed value of the diameter, the following two formulae have been employed :

$$C=\sqrt{10\,d^2},\quad A=\tfrac{1}{4}\,Cd,$$

where d=the diameter of a circle, C=its circumference and A=its area.

Umāsvāti (*c.* 150 B.C. or A.D.) writes :

"The square-root of ten times the square of the diameter of a circle is its circumference. That (circumference) multiplied by a quarter of the diameter (gives) the area".[1]

So does also Jinabhadra Gaṇi (529-589).[2]

Āryabhaṭa I says :

"Half the circumference multiplied by the semi-diameter certainly gives the area of a circle".[3]

Brahmagupta :

"Three times the diameter and the square of the semidiameter give the practical values of the circumference and area (respectively). The square roots of ten times the squares of them are the neat values".[4]

Śrīdhara :

"The square-root of the square of the diameter of a circle as multiplied by ten is its circumference. The square-root of ten times the square of the square of the semi-diameter is the area".[5]

Mahāvīra :

"Thrice the diameter is the circumference. Thrice the square of the square of the semi-diameter is the area.... So said the teachers".[6]

"The diameter of a circle multiplied by the square-root of ten, becomes the circumference. The circumference multiplied by the fourth part of the diameter gives the area".[7]

Āryabhaṭa II :

"The square-root of the square of the diameter of a circle as multiplied by ten is the circumference. The fourth part of the square of the diameter being squared and multiplied by ten, the square-root of the product is the area".[8]

---

1. *Tattvārthādhigama-sūtra* with the *Bhāṣya* of Umāsvāti, edited by K.P. Mody, Calcutta, 1903, iii. 11 (Gloss) ; *Jambūdvīpa-samāsa*, ch. iv. The latter work of Umāsvāti has been published in the Appendix C of Mody's edition of the former.
2. *Vṛhat Kṣetra-samāsa* of Jinabhadra Gaṇi, Bhavanagara, 1919, i, 7.
3. *A*, ii. 7.
4. *BrSpSi*, xii. 40.
5. *Triś*, R. 45.
6. *GSS*, vii. 19.
7. *GSS*, vii. 60.
8. *MSi*, xv. 88.

"The diameter multiplied by 22 and divided by 7 will become nearly equal to the circumference. If the square of the semi-diameter be so treated, the result will be the value of the area as precise as that of the circumference."[1]

"Twice the sine of three signs of the Zodiac (i. e. 3438) is the diameter and the circumference is then 21600. Multiply the circumference by 191 and divide by 600; the quotient is the diameter."[2]

Śrīpati's rule is the same as the first one of Āryabhaṭa the Younger Bhāskara II writes:

"When the diameter is multiplied by 3927 and divided by 1250, the result is the nearly precise value of the circumference; but when multiplied by 22 and divided by 7, it is the gross circumference which can be adopted for practical purposes."[3]

"In a circle, the one-fourth of the diameter multiplied by the circumference gives the area."[4]

"The square of the diameter being multiplied by 3927 and divided by 5000 gives the nearly precise value of the area; or being multiplied by 11 and divided by 14 gives the gross area which can be applied in rough works."[5]

Values of π

The formulae of Baudhāyana, noted above, yield the following values of π:

$$\pi = \frac{4}{\{1+\frac{1}{3}(\sqrt{2}-1)\}^2} = 3.0883\ldots\ldots$$

$$\pi = 4\left(1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8.}\right) = 3.0885\ldots$$

$$\pi = 4(1-\tfrac{2}{15})^2 = 3.004.$$

Baudhāyana has once employed the very rough value, 3. From the rule of of Manu, we get

$$\pi = 4(\tfrac{8}{9})^2 = 3.16049\ldots$$

With the corrections of Dvārakānātha, we have

$$\pi = 3.141109\ldots,\ 3.157991\ldots$$

In the early canonical works of the Jainas (500–300 B.C.) is employed the value $\pi = \sqrt{10}$.[6] This value has been adopted by Umāsvāti, Varāhamihira

1. *MSi*, xv. 92f.
2. *MSi*, xvi. 37.
3. *L*, p. 54.
4. *L*, p. 55.
5. *L*, p. 56f.
6. See Datta, Bibhutibhusan, "The Jaina School of Mathematics", *BCMS* xxi (1929), p. 13 ; "Hindu Values of π", *JASB*, xxii (1926), pp. 25-43. The latter article gives fuller informations on the subject.

(505), Brahmagupta (628), Śrīdhara (*c.* 900) and others. It is stated in the *Jīvābhigama-sūtra*,[1] that for an increment of 100 *yojanas* in the diameter, the circumference increases by 316 *yojanas*. Here has been used the value $\pi=3.16$.

Āryabhaṭa the Elder (499) gives a remarkably accurate value. His rule is :

"100 plus 4, multiplied by 8, and added to 62000 : this will be the nearly approximate (*āsanna*) value of the circumference of a circle of diameter 20000."[2]

That is to say, we have

$$\pi=\frac{62832}{20000}=\frac{3927}{1250}=3.1416.$$

This value appears in the works of Lalla[3] (*c.* 749), Bhaṭṭotpala[4] (966), Bhaskara II and others. We have it on the authority of a writer of the sixteenth century who was in possession of the larger treatise of arithmetie by Śrīdhara that this value of π was adopted there.

The value $\pi=\frac{21600}{6876}=\frac{600}{191}=3.14136\,..$

introduced first by Āryabhaṭa the Younger (950) is undoubtedly derived from the value of the Elder Āryabhata. For if the circumference of a circle measures 21600, its diameter will be

$$21600\times\frac{1250}{3927}=6875\,\frac{625}{1309}$$

Āryabhaṭa takes the value of the diameter to be 6876 in round numbers.[5] This relation (21600 : 6876) between the circumference and diameter of a circle was, however, worked out before by Bhāskara I (629).[6] The value $\pi=600/191$ appears also in the treatises of arithmetic by Gaṇeśa II (*c.* 1550) and Munīśvara (1656).

It should be particularly noted that the Greek value, $\pi=22/7$, is found in India first in the work of Āryabhaṭa the Younger[7]. Bhāskara II (1150) employs it as a rough approximation suitable for practical purposes.

---

1. *Sūtra* 112.
2. *Ā*, ii. 10.
3. *ŚiDVṛ*, i. 1, 2 ; ii. 3 ; etc.
4. See his commentary on *Bṛhat Saṃhitā*, p. 53.
5. *MSi*, xv. 88.
6. Vide his commentary on *Ā*, ii. 10.
7. *MSi*, xv. 92f.

Later Approximations of π

Later Hindu writers found much closer approximations to the value of π. Nārāyaṇa, a priest of Travancore, gave in 1426, the following rule to construct a temple of circular shape having a given perimeter:

"Divide the given perimeter into 710 parts; with 113 of them as the radius describe a circle and thus construct the circular temple".[1]

Hence he has employed $\pi=\frac{355}{113}$, the Chinese value.

Śaṅkara Vāriyar (c. 1500 - 60) says:

"The value of the given diameter being multiplied by 104348 and divided by 33215, becomes the accurate value of the circumference. Again from the circumference can be obtained the correct value of the diameter by proceeding reversely; that is, by multiplying the value of the circumference by 33215 and then dividing by 104348, or by multiplying by 113 and dividing by 355."[2]

$$\pi=\frac{104348}{33215}=3.14159265391...$$

$$\pi=\frac{355}{113}=3.1415929..$$

The first value is correct up to the ninth place of decimals, the tenth being too large, and the second up to the sixth place of decimals, the seventh being too large.

Mādhava (of Saṅgamagrāma) writes:

"It has been stated by learned men that the value of the circumference of diameter 900000000000 in length is 2827433388233".[3]

Therefore we have

$$\pi=\frac{2827433388233}{900000000000}=3.141592653592...$$

correct up to the tenth place of decimals, the eleventh being too large.

Putumana Somayāji (c. 1660-1740), the author of the *Karaṇapaddhati*, observes:

"When the value of the circumference of a circle is multiplied by 10000000000 and divided by 31415926536, the quotient is the value of the diameter. Half that is the radius."[4]

1. Nārāyaṇa, *Tantra-samuccaya*, edited by T. Gaṇapati Sastri, Trivandrum Sanskrit Series, 1919, ii. 65.
2. *Tantra-saṃgraha*, (commentary in verse, edited by K. V, Sarma), p. 103, vss. 298-9.
3. Quoted by Nīlakaṇṭha (c. 1500) in his commentary on the *Āryabhaṭīya* (ii. 10) edited by K. Sambasiva Sastri, Trivandrum Sanskrit Series, 1930.
4. *Karaṇa-paddhati*, vi. 7.

Śaṅkaravarman (1800-38) says :

"In this way, if the diameter of a great circle measure one *parārdha* (i. e. 100000000000000000), its circumference will be 314159265358979324."[1]

Here we have a value of π, 3.14159265358979324, which is correct up to 17 places of decimals.

Values in Series

Śaṅkara Vāriyar (*c.* 1500-60) gave certain interesting approximations in series for the value of the circumference of a circle in terms of its diameter. He says :

"Multiply the diameter by four and divide by one ; subtract from and add to the result alternately the successive quotients of four times the diameter divided severally by the odd numberes 3, 5, etc. Take the even number next to that odd number on division by which this operation is stopped ; then as before multiply four times the diameter by the half of that and divide by its square plus unity. Add the quotient thus obtained to the series in case its last term is negative ; or subtract if the last term be positive. The result will be very accurate if the division be continued to many terms."[2]

That is to say, if C denotes the circumference and d the diameter, then we shall have

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \ldots\ldots + (-1)^n \frac{4d}{2n+1} - (-1)^n \frac{4d\,(n+1)}{(2n+2)^2+1},$$

where $n = 1, 2, 3$ ..

He then continues :

"Now I shall write of certain other correction more accurate than this : In the last term the multiplier should be the square of half the even number together with one, and the divisor four times that, added by unity, and then multiplied by half the even number. After division by the odd numbers 3, 5, etc., the final operation must be made as just indicated."[3]

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \ldots + (-1)^n \frac{4d}{2n+1} - (-1)^n \frac{4d\,(n^2+2n+2)}{(n+1)(4n^2+8n+9)}.$$

The author seems to have realized the slow convergence of the above infinite series ; so in order to get a closer approximation to its value after

1. *Sadratna-mālā*, iv. 2.
2. *Tantra-saṃgraha*, (commentary in verse), p. 101, vss. 271-4. This rule is really that of Mādhava. See *Kriyākramakarī* (Śaṅkara Vāriyar's commentary on *Līlāvatī*), p. 379.
3. *Tantra-saṃgraha*, (commentary in verse), p. 103, vss. 295-296.

retaining a sufficient number of terms, modified the next one in the way described above and then neglected the rest. This series, without the correction in any form, is found also in the *Karaṇa-paddhati*, as follows:

"Divide four times the diameter many times severally by the odd numbers 3, 5, 7, etc. Subtract and add successive quotients alternately from and to four times the diameter. The result is an accurate value of the circumference."[1]

It was rediscovered in Europe two centuries later by Liebnitz (1673) and De Lagney (1682).

Saṅkara Vāriyar (*c*. 1500-60) says:

"The square-root of twelve times the square of the diameter is the first result. Divide this by three; again the quotient by three; and so on continuously up to as many times as desired. Then divide the results successively by the odd numbers 1, 3, etc. Of the quotients thus obtained the sum of the odd ones (i. e. 1st, 3rd, 5th,...) diminished by the sum of the even ones (i. e. 2nd, 4th, etc) will be the value of the circumference."[2]

That is to say, we shall have

$$C=\sqrt{12d^2}\left(1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.3^3}+\ldots\ldots\right).$$

The same series is described in slightly different words in the *Sadratnamālā*.[3] It is also given by Abraham Sharp (*c*. 1717), who used it for calculating the value of π up to 72 places of decimals.

Saṅkara Vāriyar writes:

"The fifth powers of the odd numbers 1, 3, etc. are increased by four times their respective roots. Divide sixteen times a given diameter severally by the sums thus obtained and subtract the sum of the even quotients from that of the odd ones. The remainder will be the circumference."[4]

That is

$$C=16d\left(\frac{1}{1^5+4.1}-\frac{1}{3^5+4.3}+\frac{1}{5^5+4.5}-\frac{1}{7^5+4.7}+\ldots\ldots\right)$$

"Or divide four times the diameter severally by the cubes of the odd numbers beginning with 3, after diminishing each by its respective root;

1. *Karaṇa-paddhati*, vi. 1.
2. *Tantra-saṃgraha* (commentary in verse), p. 96, vss. 212 (c—d)—214 (a—b).
3. *Sadratna-mālā*, iv. 2
4. *Tantra-saṃgraha* (commentary in verse), p. 102, vss. 287-8.

add and subtract the successive quotients alternately to and from thrice the diameter. Hence deduce the value of the circumference also in this way."[1]

$$C=3d+4d\left(\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\ldots\right)$$

This infinite series is stated also in the *Karaṇa-paddhati.* [2]

"Or the squares of the even numbers 2, etc. each diminished by unity, are the several denominators. Add and subtract the quotients alternately to and from twice the diameter. Take the odd number next to last even number (at which the series is stopped). The square of it added by two and then multiplied by two should be taken as the divisor at the end."[3]

$$C = 2d + 4d\left\{\frac{1}{2^2-1}-\frac{1}{4^2-1}+\ldots\ldots+(-1)^{n-1}\frac{1}{(2n)^2-1} -(-1)^{n-1}\frac{1}{2\{(2n+1)^2+2\}}\right\}$$

"Squares of the numbers beginning with two or four and increasing by four, diminished each by unity, are the several denominators; and the numerator in each case is eight times the given diameter. The value of the circumference of the circle is equal in the first case to the sum of the quotients and in the second to half the numerator minus the quotients."[4]

$$C = \left(\frac{8d}{2^2-1}+\frac{8d}{6^2-1}+\frac{8d}{10^2-1}+\ldots\ldots\right),$$

$$C = 4d-\left(\frac{8d}{4^2-1}+\frac{8d}{8^2-1}+\frac{8d}{12^2-1}+\ldots\ldots\right)$$

The *Karaṇa-paddhati* adds a new series. It says :

"Or divide six times the diameter by squares of twice the squares of even numbers minus unity as diminished by the squares of the respective even numbers. Thrice the diameter added by these quotients is the value of the circumference."[5]

$$C = 3d+\frac{6d}{(2.2^2-1)^2-2^2}+\frac{6d}{(2.4^2-1)^2-4^2}+\frac{6d}{(2.6^2-1)^2-6^2}+\ldots\ldots$$

Or,

$$C=3d+6d\left(\frac{1}{1.3.3.5}+\frac{1}{3.5.7.9}+\frac{1}{5.7.11.13}+\ldots\ldots\right)$$

---

1. *Tantra-saṃgraha* (commentary in verse), p. 103, vs. 290.
2. *Karaṇa-paddhati*, vi. 2.
3. *Tantra-saṃgraha* (commentary in verse), p. 103, vs. 292.
4. *Tantra-saṃgraha* (commentary in verse), p. 103, vss. 293-4.
5. *Karaṇa-paddhati*, vi. 4.

Sañkaravarman gives another :

"Take the square-root of twelve times the square of the diameter and also its third part. Divide these continuously by nine. Again divide the quotients (thus obtained) respectively by twice the odd numbers 1, etc. (in the former case) and by twice the even numbers 2, etc. (in the latter case), each as diminished by unity. The difference of the two sums of the final quotients is the value of the circumference of the circle."[1]

$$C = \sqrt{12d^2}\left\{\frac{1}{9(2.1-1)}+\frac{1}{9^2(2.3-1)}+\frac{1}{9^3(2.5-1)}+\dots\dots\right\}$$

$$-\frac{\sqrt{12d^2}}{3}\left\{\frac{1}{9(2.2-1)}+\frac{1}{9^2(2.4-1)}+\frac{1}{9^3(2.6-1)}+\dots\dots\right\}$$

8. Measurement of Segment of Circle

Data in Jaina Canonical Works :

In the early cosmographical works of the Jainas, we find certain interesting and valuable data relating to the mensuration of a segment of a circle.[2] Jainas suppose that Jambūdvīpa, which has been described before to be a circle of diameter 100000 *yojanas* ; is divided into seven *varṣas* ("countries") by a system of six parallel mountain ranges running due East-to-West. The southern region of it is called Bhāratavarṣa. Dimensions of this segment, in terms of *yojanas*, are as follows :

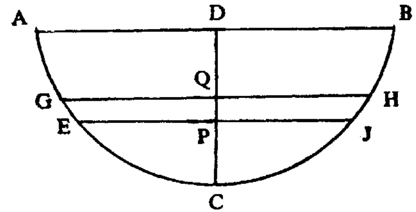


Fig. 15

$AB = 14471\frac{6}{19}$ (a little less),

$PQ = 50$,

$CD = 526\frac{6}{19}$,

$CP = QD = 238\frac{3}{19}$,

$EJ = 9748\frac{12}{19}$,

$GH = 10720\frac{12}{19}$,

$ACB = 14528\frac{11}{19}$,

$GCH = 10743\frac{15}{19}$,

$ECJ = 9766\frac{1}{19}$ (a little over),

$AG = BH = 1892\frac{7}{19} + \frac{1}{38}$,

$EG = JH = 488\frac{16}{19} + \frac{1}{38}$,

---

1. *Sadratna-mālā*, iv. 1.
2. See the article of Datta, Bhibhutibhusan, on "Geometry in the Jaina Cosmography" in *Quellen und Studien zur Gesch. d. Math*, Ab. B, Bd. 1, 1930 pp. 245-254, from which extracts are here made.

These numerical data will be found to conform to the following formulæ for the mensuration of a segment of a circle :

$$c = \sqrt{4h(d-h)},$$

$$d = \frac{c^2}{4h} + h,$$

$$a = \sqrt{6h^2 + c^2},$$

$$a' = \tfrac{1}{2}\{\text{(bigger arc)} - \text{(smaller arc)}\},$$

$$h = \tfrac{1}{2}(d - \sqrt{d^2 - c^2}),$$

or

$$h = \sqrt{(a^2 - c^2)/6},$$

where d=the diameter of the circle, c=a chord of it, a=an arc cut off by that chord, h=height of the segment or its arrow and a'=an arc of the circle lying between two parallel chords.

These formulae are not found clearly defined in abstract in any of the early canonical works, though they state in minute details some of the above numerical data.[1]

Umāsvāti's Rules

In his gloss on his own treatise *Tattvārthādhigama-Sūtra*, Umāsvāti (*c.* 150 B. C. or A. D.) says :

"The square-root of four times the product of an arbitrary depth and the diameter diminished by that depth is the chord. The square-root of the difference of the squares of the diameter and chord should be subtracted from the diameter : half of the remainder is the arrow. The square-root of six times the square of the arrow added to the square of the chord (gives) the arc. The square of the arrow plus the one-fourth of the square of the chord is divided by the arrow : the quotient is the diameter. From the northern (meaning the bigger) arc should be subtracted the southern (meaning the smaller) arc : half of the remainder is the side (arc)".[2]

All these rules have been restated by Umāsvāti in another work, *Jambūdvīpa-samāsa* by name.[3] But there the formula for the arrow is different :

"The square-root of one-sixth of the difference between the squares of the arc and the chord is the arrow."

It is clearly approximate.

1. For instance see *Jambūdvīpa-prajñapti, Sūtra* 3, 10-15 ; *Jīvābhigama-sūtra, Sūtra* 82, 124; *Sūtrakṛtāṅga-sūtra, Sūtra*, 12.
2. *Tattvārthādhigama-sūtra*, iii. 11 (Gloss).
3. *Jambūdvīpa-samāsa*. ch. iv.

Āryabhaṭa I and Brahmagupta

Āryabhaṭa I writes :

"In a circle, the product of the two arrows is the square of the semi-chord of the two arcs."[1]

Brahmagupta says :

"In a circle, the diameter should be diminished and then multiplied by the arrow ; then the result is multiplied by four : the square root of the product is the chord. Divide the square of the chord by four times the arrow and then add the arrow to the quotient : the result is the diameter. Half the difference of diameter and the square-root of the difference between the squares of the diameter and chord, is the smaller arrow."[2]

Jinabhadra Gaṇi's Rules. Jinabhadra Gaṇi (529-589) writes :

"Multiply by the depth, the diameter as diminished by the depth : the square-root of four times the product is the chord of the circle."[3]

"Divide the square of the chord by the arrow multiplied by four ; the quotient together with the arrow should be known certainly as the diameter of the circle. The square of the arrow multiplied by six should be added to the square of the chord ; the square-root of the sum should be known to be the arc. Subtract the square of the chord certainly from the square of the arc ; the square-root of the sixth part of the remainder is the arrow. Subtract from the diameter the square-root of the difference of the squares of the diameter and chord ; half the remainder should be known to be the arrow."[4]

"Subtract the smaller arc from the bigger arc ; half the remainder should be known to be the side arc. Or add the square of half the difference of the two chords to the square of the perpendicular ; the square-root of the sum will be the side arc."[5]

Jinabhadra Gaṇi next cites two formulae for finding the area of a segment of a circle cut off by two parallel chords.

"For the area of the figure, multiply half the sum of its greater and smaller chords by its breadth."[6]

1. *Ā*, ii. 17.
2. *BrSpSi*, xii. 41 f.
3. *Vṛhat Kṣetra-samāsa* of Jinabhadra Gaṇi, i. 36.
4. *Vṛhat Kṣetra-samāsa*, i. 38-41.
5. *Vṛhat Kṣetra-samāsa*, i. 46-7.
6. *Ibid*, i. 64.

or

"Sum up the squares of its greater and smaller chords ; the square-root of the half of that (sum) will be the 'side'. That multiplied by the breadth will be its area."[1]

That is to say, if c_1, c_2 be the lengths of the two parallel chords and h, the perpendicular distance between them, then the area of the segment will be given by

(i) $\text{Area} = \frac{1}{2}(c_1 + c_2)\, h$

(ii) $\text{Area} = \sqrt{\tfrac{1}{2}(c_1^2 + c_2^2)} \times h.$

Neither of these formulae, the author thinks, will be available for finding the area of the Southern Bhāratavarṣa which, as has been described before, has only a single chord. So he gives a third formula as follows :

"In case of the Southern Bhāratavarṣa, multiply the arrow by the chord and then divide by four; then square and multiply by ten : the square-root (of the result) will be its area."[2]

(iii) $$\text{Area} = \sqrt{10\left(\frac{ch}{4}\right)^2}.$$

None of the above formulae will give the desired result to a fair degree of accuracy. Formula (i) indeed gives the area of the isosceles trapezium of which the two parallel chords form the two parallel sides. The result obtained by it will therefore be approximately correct only when the breadth is small. Otherwise as has been observed by the commentator Malayagiri (*c.* 1200), the formula will give only a wrong result. Jinabhhadra Gaṇī seems to have been aware of this limitation of the formula. For he has not followed it in practice. The rationale of formula (ii) which has been followed by our author, cannot be easily determined. Formula (iii) seems to have been derived by analogy with the formula for finding the area of a semi-circle.

Śrīdhara's Rule.

In his smaller treatise of arithmetic, Śrīdhara (*c.* 900) includes a formula for finding the area of a segment of a circle. He says :

"Multiply half the sum of the chord and arrow by the arrow ; multiply the square of the product by ten and then divide by nine. The square-root of the result will be the area of the segment."[3]

$$\text{Area} = \sqrt{\frac{10}{9}\left\{h\left(\frac{c+h}{2}\right)\right\}^2}.$$

1. *Vṛhat Kṣetra-samāsa*, i. 122.

2. *Ibid.*, i. 122.

3. *Triś*, R. 47.

Mahāvīra's Rules

For the mensuration of a segment of a circle, Mahāvīra (850) gives two sets of formulae ; the first set gives results serving all practical purposes (*vyāvahārika phala*), while the second set yields nearly precise results (*sūkṣma phala*). He says :

"Multiply the sum of the arrow and chord by the half of the arrow : the product is the area of the segment. The square-root of the square of the arrow as multiplied by five and added by the square of the chord is the arc."[1]

"The square-root of the difference between the squares of the arc and chord, as divided by five, is stated to be the arrow. The square-root of the square of the arc minus five times the square of the arrow is the chord."[2]

Thus the rough formulae are :

$$\text{Area} = \tfrac{1}{2}\, h\,(c+h),$$

$$h = \sqrt{\frac{a^2 - c^2}{5}}\ ,$$

$$c = \sqrt{a^2 - 5h^2}\ ,$$

$$a = \sqrt{5h^2 + c^2}\ .$$

For calculation of nearly precise results his rules are as follow :

"In case of a figure of the shape of (the longitudinal section of) a barley and a segment of a circle, the chord multiplied by one-fourth the arrow and also by the square-root of ten becomes, it should be known, the area."[3]

"The square of the arrow is multiplied by six and then added by the square of the chord ; the square-root of the result is the arc. For finding the arrow and the chord the process is the reverse of this. The square-root of the difference of the squares of the arc and chord, as divided by six, is stated to be the arrow. The square-root of the square of the arc minus six times the square of the arrow is the chord."[4]

Thus the nearly precise formulae of Mahāvīra are :

$$\text{Area} = \frac{\sqrt{10}}{4}\, ch,$$

$$h = \sqrt{(a^2 - c^2)/6}\ ,$$

$$a = \sqrt{6h^2 + c^2}\ ,$$

$$c = \sqrt{a^2 - 6h^2}\ .$$

1. *GSS*, vii. 43. 2. *GSS*, vii. 45. 3. *GSS*, vii. 70½. 4. *GSS*, vii. 74½.

Āryabhaṭa II's Rules

Like Mahāvīra, Āryabhaṭa II (950) too gives two sets of formulae, rough (*sthūla*) as well as nearly precise (*sūkṣma*) for the mensuration of a segment of a circle. But it will be noticed that the rough formulae are the same as the nearly precise ones of his predecessor; one about the area yields distinctly better results. Āryabhaṭa II writes:

"The product of the arrow and half the sum of the chord and arrow is multiplied by itself; the square-root of the result increased by its one-ninth is the rough value of the area of the segment. The square-root of the square of the arrow multiplied by six and added by the square of the chord is the arc. The square-root of the difference of the square of the arc and chord as divided by six, is the arrow. The square-root of the remainder left on subtracting six times the square of the arrow from the square of the arc, is the chord. The half of the arc multiplied by itself is diminished by the square of the arrow; on dividing the remainder by twice the arrow, the quotient will be the value of the diameter."[1]

That is to say, the rough formulae are:

$$\text{Area}=\sqrt{(1+\tfrac{1}{9})\left\{h\left(\frac{c+h}{2}\right)\right\}^2}$$

$$a=\sqrt{6h^2+c^2},$$

$$h=\sqrt{(a^2-c^2)/6},$$

$$c=\sqrt{(a^2-6h^2)},$$

$$d=\frac{1}{2h}\,(\tfrac{1}{2}\,a^2-h^2).$$

Āryabhaṭa II then continues:

"On dividing by 21 the product of half the sum of the chord and arrow, as multiplied by the arrow and again by 22, the quotient will be the nearly precise value of the area of the segment. The square of the arrow being multiplied by 288 and divided by 49, is increased by the square of the chord; the square-root of the result is the near value of the arc. The square-root of the difference of the squares of the arc and chord, as multiplied by 49 and divided by 288, is the arrow. The square-root of what is left on subtracting from the square of the arc, the square of the arrow multiplied by 288 and divided by 49 will be the chord. Multiply the square of the arc by 245 and then divide by 484; divide the quotient as diminished by the square of the arrow, by twice the arrow: the quotient will be the

1. *MSi*, xv. 89-92.

diameter. Similarly the chord will be the square-root of the diameter as diminished by the arrow and then multiplied by four times the arrow. The square-root of the difference of the squares of the diameter and chord being subtracted from the diameter, half the remainder is the arrow. The square of the semi-chord being added with the square of the arrow, the quotient of the sum divided by the arrow is the diameter."[1]

Hence

$$\text{Area} = \frac{22}{21}h\left(\frac{c+h}{2}\right),$$

$$a = \sqrt{\frac{288}{49}h^2 + c^2},$$

$$h = \sqrt{\frac{49}{288}\left(a^2 - c^2\right)},$$

$$c = \sqrt{a^2 - \frac{288}{49}h^2},$$

$$d = \frac{1}{2h}\left(\frac{245}{484}a^2 - h^2\right),$$

$$e = \sqrt{4h(d-h)},$$

$$h = \tfrac{1}{2}\{d - \sqrt{d^2 - c^2}\},$$

$$d = \frac{1}{h}\left\{\left(\frac{c}{2}\right)^2 + h^2\right\}.$$

It should perhaps be noted that the last three formulae are exact, while others are approximate.

Śrīpati's Rules

Śrīpati (*c.* 1039) states :

The diameter of a circle is diminished by the given arrow and then multiplied by it and also by four : the square-root of the result is the chord. In a circle, the square-root of the difference of the squares of the diameter and chord being subtracted from the diameter, half the remainder is the arrow. In a circle, the square of the semi-chord being added to the square of the arrow and then divided by the arrow, the result is stated to be the diameter.........Six times the square of the arrow being added to the square of the chord, the square-root of the sum is the arc here. The difference of the squares of the arc and chord being divided by six, the square-root of the quotient is the value of the arrow. From the square of the arc being subtracted the square of the arrow as multiplied by six, the square-root of

1. *MSi*, xv. 93-99.

the remainder is the chord. Twice the square of the arrow being subtracted from the square of the arc, the remainder divided by four times the arrow, is the diameter."[1]

Bhāskara II's Rules

Bhāskara II (1150) does not mention the formulae for the calculation of approximate results, but gives only the exact ones. He writes:

"Find the square-root of the product of the sum and difference of the diameter and chord, and subtract it from the diameter: half the remainder is the arrow. The diameter being diminished and then multiplied by the arrow, twice the square-root of the result is the chord. In a circle, the square of the semi-chord being divided and then increased by the arrow, the result is stated to be the diameter."[2]

These rules have been reproduced by Munīśvara.[3]

Sūryadāsa's Proof

Sūryadāsa (born 1508) proves the formulae for the arrow and diameter as follows:

Let AB be a chord of the circle having its centre at O and CH the arrow of the segment ABC. Join BO and produce it to meet the circumference in P. Draw PSQ parallel to AB. Join BQ. Then clearly

$$CH = \tfrac{1}{2}(CR - HS)$$
$$= \tfrac{1}{2}(CR - BQ)$$
$$= \tfrac{1}{2}(CR - \sqrt{BP^2 - PQ^2}).$$

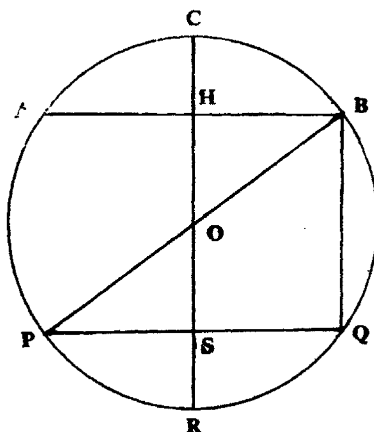


FIG. 16.

Hence

$$CH = \tfrac{1}{2}(CR - \sqrt{CR^2 - AB^2}).$$

Again, since

$$HB^2 = CH.HR,$$

we get

$$HR = \frac{HB^2}{CH}$$

Therefore

$$CR = \frac{HB^2}{CH} + CH.$$

1. *SiŚe*, xiii. 37-40,
2. *L*, p. 58.
3. *Pāṭī-sāra*, R. 220-1.

Other formulae for Area

For the area of a segment of a circle, Viṣṇu Paṇḍita (*c.* 1410) and Keśava II (1496) gave the formula :

$$\text{Area} = \left(1 + \frac{1}{20}\right) \left\{h\left(\frac{h+c}{2}\right)\right\}$$

Gaṇeśa (1545) and Rāmakṛṣṇadeva state :

Area = (area of the sector) − (area of the triangle)

$= \frac{1}{4}\, ad - \frac{1}{2} c\, (\frac{1}{2}d - h).$

Intersection of two circles

When two circles intersect, the common portion cut off is called the *grāsa* ("the erosion"). The origin of the term seems to be connected with the eclipse of the moon (or the sun) which is narrated in the popular mythology of the early Hindus as being caused by the dragon *Rāhu* (earth's shadow) swallowing the moon. The portion swallowed up is the *grāsa*. In fact, the geometrical theorem, just to be described, had its application in the calculation of the eclipse. The common portion is also called *matsya* (fish) as it resembles a fish.

Āryabhaṭa I writes :

"(The diameters of) the two circles being severally diminished and then multiplied by (the breadth of) the erosion, the products divided severally by the sum of the diameters (each) as diminished by the erosion, will be the two arrows lying within the erosion."[1]

This rule is nearly reproduced by Mahāvīra.[2]

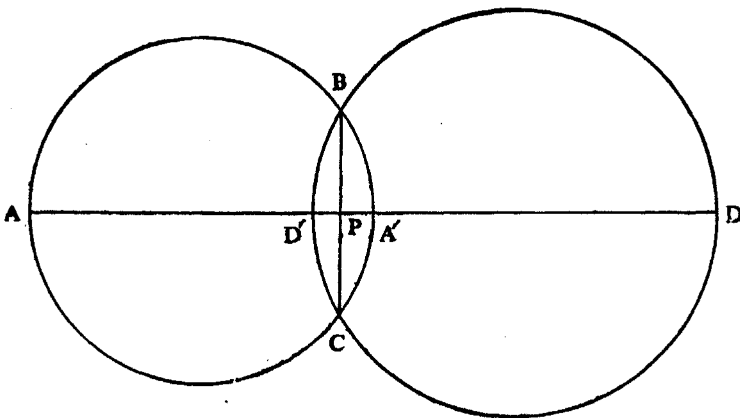


Fig. 17.

1. *Ā*, ii. 18.

2. *GSS*, vii. 231½.

$$AP.PA' = PB^2 = DP.PD',$$

or $$(AA' - A'P)\, A'P = (DD' - D'P)\, D'P,$$

or $$AA'.A'P - DD'.D'P = A'P^2 - D'P^2$$
$$= (A'P + D'P)\,(A'P - D'P)$$
$$= A'D'\,(A'P - D'P),$$

or $$(AA' - A'D')\, A'P = (DD' - A'D')\, D'P.$$

Hence

$$\frac{A'P}{DD' - A'D'} = \frac{D'P}{AA' - A'D'}$$
$$= \frac{A'D'}{(DD' - A'D') + (AA' - A'D')}$$

Therefore

$$A'P = \frac{A'D'\,(DD' - A'D')}{DD' + AA' - 2A'D'}$$

$$D'P = \frac{A'D'\,(AA' - A'D')}{DD' + AA' - 2A'D'}.$$

Brahmagupta says :

"The erosion being subtracted (severally) from the two diameters, the remainders, multiplied by the erosion and divided by the sum of the remainders, are the arrows."[1]

"The square of half the (common) chord being divided severally by the two given arrows, the quotients added with the respective arrows give the two diameters. The sum of the two arrows is the erosion ; and that of the quotients is the sum of the diameters minus the erosion."[2]

9 Miscellaneous Figures

Miscellaneous Figures

Śrīdhara, Mahāvīra and Āryabhaṭa II have treated the mensuration of certain other plane figures such as of the shape of a barley corn (*yava*), drum (*muraja*, *mṛdaṅga*), elephants's tusk (*gajadanta*), crescent moon (*bālendu*), felloe (*nemi*), thunder-bolt (*vajra*) etc. The formulae described in case of most of them are only roughly approximate and some of them are deduced easily from the results already obtained. It was probably from the point of view of some practical utility that all the results have been stated separately.

1. *BrSpSi*, xii. 42.
2. *BrSpSi*, xii. 43.

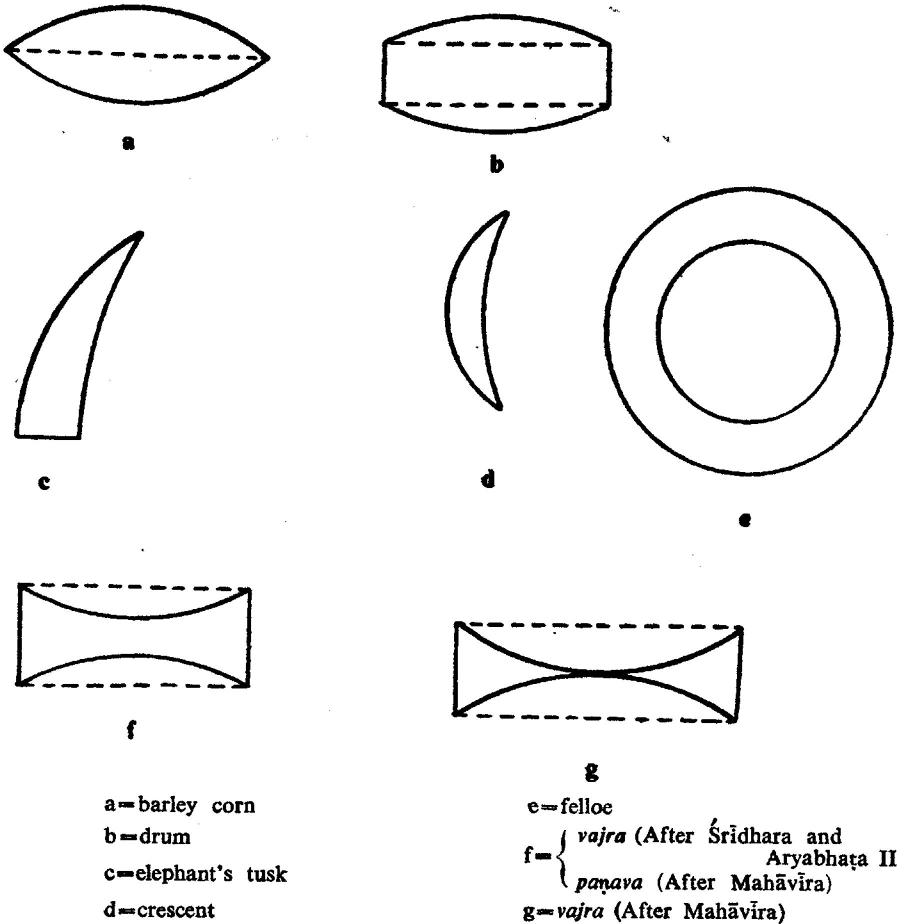


FIG. 18

Śrīdhara's Rules. Śrīdhara says:

"A figure of the shape of an elephant tusk (may be considered) as a triangle, of a felloe as a quadrilateral, of a crescent moon as two triangles and of a thunderbolt as two quadrilaterals."[1]

"A figure of the shape of a drum, should be supposed as consisting of two segments of a circle with a rectangle intervening; and a barley corn only of two segments of a circle."[2]

Mahāvīra's Rules. For finding the gross value of the areas of above figures Mahāvīra gives the following rules:

1. *Triś,* R. 44.
2. *Triś,* R. 48.

"In a figure of the shape of a felloe, the area is the product of the breadth and half the sum of the two edges. Half that area will be the area of a crescent moon here."[1]

"The diameter increased by the breadth of the annulus and then multiplied by three and also by the breadth gives the area of the outlying annulus. The area of an inlying annulus (will be obtained in the same way) after subtracting the breadth from the diameter."[2]

"In case of a figure of the shape of a barley corn, drum, *paṇava*, or thunderbolt, the area will be equal to half the sum of the extreme and middle measures multiplied by the length."[3]

For finding the neat values of the areas of them, Mahāvīra has the following rules :

"The diameter added with the breadth of the annulus being multiplied by $\sqrt{10}$ and the breadth gives the area of the outlying annulus. The area of the inlying annulus (will be obtained from the same operations) after subtracting the breadth from the diameter."[4]

"Find the area by multiplying the face by the length. That added with the areas of the two segments of the circle associated with it will give the area of a drum-shaped figure. That diminished by the areas of the two associated segments of the circle will be the area in case of a figure of the shape of a *paṇava* as well as of a *vajra*."[5]

"In case of a felloe-shaped figure, the area is equal to the sum of the outer and inner edges as divided by six and multiplied by the breadth and $\sqrt{10}$. The area of a crescent moon or elephant's tusk is half that."[6]

Āryabhaṭa II's Rules.

Āryabhaṭa II writes :

"In (a figure of the shape of) the crescent moon, there are two triangles and in an elephant's tusk only one triangle ; a barley corn may be looked upon as consisting of two segments of a circle or two triangles."[7]

1. *GSS*, vii. 7. The formula for the area of the felloe yields, indeed, the accurate value of it.
2. *GSS*, vii. 28.
3. *GSS*, vii, 32.
4. *GSS*. vii. 67½.
5. *GSS*, vii. 76½.
6. *GSS*, vii. 80½.
7. *MSi*, xv. 101.

"In a drum, there are two segments of a circle outside and a rectangle inside; in a thunderbolt, are present two segments of two circles and two quadrilaterals."[1]

Polygons

According to Śrīdhara, regular polygons may be treated as being composed of triangles.[2] Mahāvīra says :

"One-third of the square of half the perimeter being divided by the number of sides and multiplied by that number as diminished by unity will give the (gross) area of all rectilinear figures. One-fourth of that will be the area of a figure enclosed by circles mutually in contact."[3]

That is to say, if $2s$ denote the perimeter of a polygon of n sides, whether regular or otherwise, but without a re-entrant angle, then its area will be roughly given by the formula

$$\text{Area} = \frac{(n-1)\, s^2}{3n}.$$

Mahāvīra has treated some very particular cases of polygons with re-entrant angles. He says :

"The product of the length and the breadth minus the product of the length and half the breadth is the area of a di-deficient figure ; by subtracting half the latter (product from the former) is obtained the area of a uni-deficient figure."[4]

The figures contemplated in this rule are those formed by leaving out two vertically opposite ones or any one of the four portions into which a rectangle is divided by its two diagonals. In the first case, the figure is technically called the *ubhaya-niṣedha-kṣetra* ("di-deficient figure") and in the other the *eka-niṣedha-kṣetra* (uni-deficient figure).

Mahāvīra further says :

"On subtracting the accurate value of the area of one of the circles from the square of a diameter, will be obtained the (neat) value of the area of the space lying between four equal circles (touching each other)."[5]

"The accurate value of the area of an equilateral triangle each side of which is equal to a diameter, being diminished by half the area of a circle, will yield the area of the space bounded by three equal circles (touching each other)."[6]

---

1. *MSi*, xv. 103.
2. *Triś*, R. 48.
3. *GSS*, vii. 39.
4. *GSS*. vii. 37.
5. *GSS*, vii. 82½.
6. *GSS*, vii. 84½.

"A side of a regular hexagon, its square and its biquadrate being multiplied respectively by 2, 3, and 3 will give in order the value of its diagonal, the square of the altitude, and the square of the area."[1]

Āryabhaṭa II observes :

"A pentagon is composed of a triangle and a trapezium, a hexagon of two trapeziums ; in a lotus-shaped figure there is a central circle and the rest are triangles."[2]

Ellipse

Though the ellipse was known to the Hindus as early as *circa* 400 B.C., we do not find any formula for its mensuration in any of their works on mathematics, except the *Gaṇita-sāra-saṃgraha* of Mahāvīra (850). In the latter again, we have only roughly approximate results. Mahāvīra says :

"The length of an ellipse being added by half its breadth and multiplied by two, gives the gross value of its circumference. The circumference multiplied by one-fourth the breadth becomes the gross value of the area."[3]

"The square-root of six times the square of the breadth added with the square of twice the length, will be the neat value of the circumference of an ellipse. That multiplied by one-fourth the breadth will become the neat value of the area."[4]

That is to say if $2a$ be the longer diameter of an ellipse and $2b$ its shorter diameter, then, according to Mahāvīra,

$$\text{Circumference } (\textit{Gross}) = 2\,(2a+b),$$
$$\text{Circumference } (\textit{Neat}) = \sqrt{16a^2+24b^2},$$
$$\text{Area } (\textit{Gross}) = b\,(2a+b),$$
$$\text{Area } (\textit{Neat}) = \tfrac{1}{2}b\sqrt{16a^2+24b^2}.$$

10. MEASUREMENT OF VOLUMES

Solids Considered

Things in everyday life of the ancient Vedic Hindus which led them to develop formulae for the measurement of volumes were fire-altars and excavations. Amongst the fire-altars described in the extant works on the *Śulba*, we find that some are right prisms of various cross-sections, and others are right circular cylinders. Only in one case, namely, the fire-altar of the shape of the cemetery, the solid considered resembles a frustum of a pyramid. For the measurement of the latter, the Hindus developed an approximate formula. Though we meet with copious descriptions of pits, caves and mountains etc., of the shape of truncated cones and pyramids, in the

1. *GSS*, vii. 86½.
2. *MSi*, xv. 102.
3. *GSS*, vii. 21.
4. *GSS*, vii. 63.

early canonical works of the Jainas, there is nothing to indicate that the mensuration of those solids was known to them. In later Hindu treatises of arithmetic, solids generally treated are excavations, mounds of grains and piles of bricks.

Prism and Cylinder

The formula for calculating the volumes of prisms and cylinders is found in the *Śulba*.[1]

Volume of a prism or cylinder = (base) × (height).

The same formula is stated in later works.[2]

It may be noted that in later treatises of arithmetic, an excavation (*khāta*) whose depth is uniform is called the *sama-khāta*. The section of the base may be of any form, as it has not been particularly mentioned. The word *sama* (equal) implies that all sections parallel to the face or base are equal.

In the *Veda* and *Saṃhitā*, the prisms whose sections are regular polygons, were named according to the number of edges. Thus in the *Ṛgveda* (*c*. 3000 B.C.), the triangular prism is called *trirasri* (three-edged solid; *tri*=three, *asri*=edge), a quadrangular prism *caturasri* (=four-edged solid) and so on[3]. But these terms do not seem to have been completely standardized. For in comparatively later times, a cube was called *dvādasāsrika* (=twelve-edged solid).

Cone and Pyramid

The Hindus do not always distinguish between a cone and a pyramid. They include both under a generic name *sūcī*, which means literally "a needle," "a sharp pointed object", and hence, "a solid of the form of the needle", "a sharp pointed solid." Thus the term generally denotes a pyramid with a base of any form; as the base may be a circle it includes a cone as well. A triangular pyramid is, however, distinguished as the *ghana ṣaḍasri* or simply *ṣaḍasri* (literally, "six-edged solid").

Āryabhaṭa I says:

"Half the product of this area (of the triangular base) and the height is the volume of the six-edged solid."[4]

This formula for the volume of the triangular pyramid is wrong. The correct formula is found in the works of Brahmagupta. He states:

1. Datta, *Śulba*, p. 101. See also *Jaina Math., Quel, und Stud. z. Gesch. d. Math*, Bd. I. (1930), p. 253.
2. *BrSpSi*, xii 44; *Triś*, R. 53; *GSS*, viii. 4. etc.
3. Datta, "On the Hindu names for the rectilinear geometrical figures", *loc. cit*, pp. 284f.
4. *Ā*, ii. 6.

"The volume of the uniform excavation divided by three is the volume of the needle-shaped solid."[1]

That is to say, we shall have

Volume of a cone or pyramid $= \frac{1}{3}$ (base) × (height).

This formula reappears in the works of Āryabhaṭa II[2], Nemicandra,[3] Śrīpati[4] and Bhāskara II[5].

For measuring the mounds of grains which approximate to the form of a right circular cone, the Hindus ordinarily employed a rough formula. In such cases, they further assume the height of the mound to be equal to the circumference of the base divided by 9, 10 or 11 according to the kind of grain of which the mound is composed. Thus Brahmagupta says :

"In case of *śuki* grains one-ninth, in case of course grains one-tenth and in case of fine grains one-eleventh of the circumference (of the base) is the height ; that multiplied by the square of the sixth part of the circumference will be the volume. "[6]

Śrīpati writes :

"Of a heap of grains standing on the plane surface of the earth, the square of one-sixth the circumference multiplied by the height is the volume in terms of Māgadha Khārikā. In case of grains known as *syāmāka*, *śāli*, *tila*, *sarṣapa*, etc. the circumference is nine times the height ; in case of *godhūma*, *mudga*, *yava*, *dhānyaka*, etc., it is ten times ; and in case of *vadara*, *kaṅgu*, *kulattha*, etc., eleven times."[7]

The rough formula was obtained probably thus :

$$\text{Volume of a cone} = \tfrac{1}{3}(\text{base}) \times (\text{height}).$$

If r denote the radius of the base, we have

$$\text{Base} = \pi r^2 = \frac{2\pi r . 2\pi r}{4\pi} = \frac{(\text{Circumference})^2}{4\pi} .$$

Hence

$$\text{Volume of a cone} = \frac{1}{12\pi}\,(\text{circumference})^2 \times (\text{height}).$$

Now putting roughly we get $\pi = 3$,

$$\text{Volume} = \left(\frac{\text{circumference}}{6}\right)^2 \times (\text{height}).$$

1. *BrSpSi*, xii. 44.
2. *MSi*, xv. 105.
3. *Trilokasāra, Gāthā* 19.
4. *SiŚe*, xiii. 44.
5. *L*, p. 62.
6. *BrSpiS*, xii. 50.
7. *SiŚe*, xiii. 50-1.

This approximate formula is stated also by Sridhara,[1] Āryabhaṭa II,[2] Nemicandra[3] and Bhāskara II.[4] The ancient commentators have observed that it was intended only for "rough calculation."

Frustum of a Cone

To find the volume of a frustum of a right circular cone, Śrīdhara gives the following formula :

"The square-root of ten times the square of the sum of the squares of the diameters of the face, base and of their sum, being multiplied by the height and divided by twenty-four, will be the volume of a well."[5]

That is to say, if d, d' denote the diameters of the upper and lower faces of the frustum of a right circular cone and h its height, then its volume V will be given by

$$V = \frac{h}{24}\sqrt{10\{d^2 + d'^2 + (d+d')^2\}^2},$$

or

$$V = \frac{\pi}{3}(r^2+r'^2+rr')\,h,$$

where r, r' denote the radii of the upper and lower faces and $\pi=\sqrt{10}$, the value adopted by Śrīdhara. Other writers have included the treatment of the frustum of a cone in that of a more general kind of obelisk.

Example from Śrīdhara : "The diameter of the top of a well is 16 cubits, and of the bottom 4 cubits ; its depth is 12 cubits. Find, O learned man, its volume."[6]

Obelisk

An approximate formula for calculating the volume of a frustum of a pyramid on a rectangular base is found as early as the works on the *Śulba* by Baudhāyana (800 B. C.) and others.[7] If (a, b) be the length and breadth of the base of the solid, (a',b') the corresponding sides of the face parallel to it and h the height then

$$\text{Volume of the frustum} = \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right) \times h.$$

In later treatises of arithmetic we find the accurate formula for the same. Thus Brahmagupta says :

"The area from half the sum of (the edges of) the face and base, being multiplied by the depth gives *Vyāvahārika* volume ; half the sum of the areas of the face and base being multiplied by the depth will be the *Autra* volume.

1. *Triś*, R. 61.
2. *MSi*, xv. 115.
3. *Trilokasāra*, *Gāthā* 22, 23.
4. *L*, pp. 69f.
5. *Triś*, R. 54.
6. *Triś*, Ex, 91.
7. Datta, *Śulba*, p. 103.

Subtract the *Vyāvahārika* volume from the *Autra* volume and divide the remainder by three ; the quotient added with the *Vyāvahārika* volume will become the truly accurate volume."[1]

It is noteworthy that Brahmagupta does not specify the shape of the face and base of the excavation contemplated by him. His text is *mukhatalayutidalagaṇitam* etc., or "the area from the half the sum of the face and base," etc. If we, however, suppose them to be rectangular, then according to the rule, we shall have,

$$V' = \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right)h,$$

$$A = \tfrac{1}{2}(ab+a'b')\,h,$$

$$V = \tfrac{1}{3}(A-V')+V',$$

where V', A and V denote respectively the *Vyāvahārika*, *Autra* and accurate volumes of the obelisk. Substituting the values in the last formula, we get

$$V = \frac{h}{6}\{(a+a')(b+b') + ab+a'b'\}.$$

If the face and base be *circular*, and if their radii be r' and r respectively, then by the rule

$$V' = \pi\left(\frac{r+r'}{2}\right)^2 h = \frac{\pi}{4}(r+r')^2 h,$$

$$A = \left(\frac{\pi r^2+\pi r'^2}{2}\right)h.$$

Hence

$$V = \frac{h}{3}\left\{\frac{\pi}{2}(r^2+r'^2) - \frac{\pi}{4}(r+r')^2\right\} + \frac{\pi}{4}(r+r')^2 h$$

$$= \tfrac{1}{3}\pi h\,(r^2+r'^2+rr').$$

Particular cases

(i) Put $a' = 0 = b'$; then we get
Volume of a cone or pyramid $= \tfrac{1}{3}$ (base) (height).

(ii) Let $b'=0$;
Volume of a wedge $= \frac{h}{6}(2ab+a'b)$.

(iii) Suppose $a=b$, $a'=b'$; then
Volume of a truncated square pyramid $= \frac{h}{3}(a^2+a'^2+aa')$.

Pṛthūdakasvāmi has worked out the following example in illustration of the above rule of Brahmagupta :

1. *BrSpSi.* xii. 45-6.

"There is a square tank whose each side is 10 cubits long at the face and 6 cubits long at the base; it is excavated so as to have a depth of 30 cubits. Tell me its *Vyāvahārika*, *Autra* and truly accurate volumes."

This example has misled some of the modern historians of mathematics to presume that Brahmagupta's rule was meant for the measurement of the volume of a truncated pyramid on a square base only.[1] But, as already pointed out, there is nothing in the definition of the rule to warrant such a limited application of it.[2]

Mahāvīra writes:

"Of the outer (i. e. at the ground) and various inner sections (of the excavation) the sides of the ground section are added by all the corresponding sides of other sections and divided (by the number of sections). Multiply these sides (of the average section) mutually in accordance with the method of finding the area of a figure of that shape; the result (thus obtained) multiplied by the depth will be the *Karmāntika* volume. Find the areas of those sections (severally), add them together and then divide by the number of sectional areas; the quotient multiplied by the depth will be the *Aundra* volume. One third the differences of those two volumes added with the *Karmāntika* volume will be the truly accurate volume."[3]

It will be noticed that in finding the average volumes, Mahāvīra takes into consideration several parallel sections of the solid, instead of only two, the face and base.[4] In three of the illustrative examples,[5] he actually states three sections of the solid. If however, we take into consideration only the top and base, the formula obtained will be the same as that of Brahmagupta.

In illustration of his rule, Mahāvīra gives examples of excavations of various kinds, which are indeed inverted cases of truncated pyramids on square, rectangular, or equilateral triangular bases, and truncated cones. There is an instance of a truncated wedge:

"(In a well with rectangular sections), the lengths at the top, middle and base are 90, 80 and 70 respectively; and the breadths are 22, 16 and 10. Its depth is 7. (Calculate its volume)." [6]

1. Such is the opinion of Cantor, followed by J. Tropfke and D. E. Smith.
2. See also the article of Datta, Bibhutibhusan "On the supposed indebtedness of Brahmagupta to Chiu-chang Suan-Shu in the *BCMS*, xiii (1930) pp. 39-51; more particularly pp. 45 ff.
3. *GSS*, viii, 9-11½.
4. Hence Raṅgācārya is wrong in supposing that the rule contemplates only the face and base.
5. *GSS*, viii, 16½—18½.
6. *GSS*, vii. 16½. In the printed text 22 is wrongly stated as 32.

Āryabhaṭa II says :

"Divide the sum of the areas of the face, base and that arising from the sum of (the dimensions of) them by six ; the quotient multiplied by the height will be the volume of an excavation such as a well and tank."[1]

That is to say,

$$V = \frac{h}{6}\{(a+a')(b+b')+ab+a'b'\}.$$

This formula reappears in the works of Śrīpati and Bhāskara II. The former says :

"The sum of the areas of the face, base and that arising from the sum of their sides, being divided by six and multiplied by the depth, will be the truly accurate value of the volume.[2]

Bhāskara II writes :

"The sum of the areas from (the linear dimensions of) the face, base and their sums, divided by six gives the area of the equivalent prism (*samaṁ kṣetraphalam*) (of the same height). That multiplied by the depth is the true volume."[3]

Gaṇeśa's Proof : Gaṇeśa demonstrates this formula substantially as follows :

Suppose (a, b) and (a', b') denote the length and breadth of the face and base of the solid respectively. Let its height be h. Then it is clear from the figure that

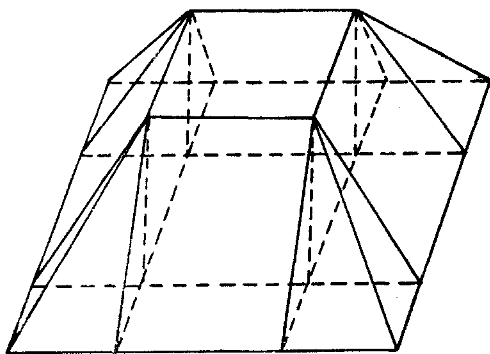


FIG. 19

Volume of the obelisk

=volume of the prism at the centre
+volmes of four pyramids at the corners
+volumes of four prisms on four sides.

Now the four pyramids at the corners can be combined into one of base $(a-a')$ by $(b-b')$ and height h. Hence its volume is

$$\frac{h}{3}(a-a')(b-b').$$

The four side prisms can be combined into two others : (1) one on a triangle of base $(b-b')$ and altitude h, its height being a' ; and (2) the other

1. *MSi*, xv. 106.
2. *Si Se*, xiii. 49.
3. *L*, p. 65.

on a triangle of base $(a-a')$ and altitude h; its height wlll be b'. Therefore their volumes are together equal to

$$\tfrac{1}{2}(b-b')\,ha'+\tfrac{1}{2}(a-a')\,hb'.$$

Therefore

Volume of the obelisk

$$= a'b'h + \frac{h}{3}(a-a')(b-b')+\tfrac{1}{2}(b-b')\,ha'+\tfrac{1}{2}(a-a')\,hb'.$$

$$= \frac{h}{6}(2ab+2a'b'+a'b+ab')$$

$$= \frac{h}{6}\{(a+a')(b+b')+ab+a'b'\}.$$

Mahāvira has treated a problem like this: A fort wall of height h and length l, whose extremities are vertical, has its base b in breadth and face a. Its upper portion is blown off by cyclone, obliquely. It is required to calculate the volume of the portion still intact.

Another problem runs as follows:

"The heights (of a certain construction) are 12, 16, and 20 cubits (at one end, middle and other end respectively); the breadths (at those points) are respectively 7, 6 and 5 cubits at the base and 4, 3 and 2 cubits at the top; the length is 24 cubits. (Find the number of bricks employed in the construction)"[3]

Surface of a Sphere

The earliest reference to a formula for the surface of a sphere occurs, so far as known, in the treatise on arithmetic by Lalla (*c.* 749). That work is now lost. But the relevant passage has survived in a citation by Bhāskara II[4]. It is as follows:

"The area of the circle (of a diametral section) multiplied by its circumference will be equal to the area of the surface of a sphere."

If r be the radius of a sphere, then according to this rule, its surface S will be

$$S=\pi r^2 \times 2\pi r = 2\pi^2 r^3.$$

This formula is clearly inaccurate. So it has been adversely criticised and discarded by Bhāskara II.[5]

1. *GSS*, viii. $52\frac{1}{2}$. 2. *GS*, viii. $54\frac{1}{2}$. 3. *GSS*, viii. $51\frac{1}{2}$
4. *SiSi, Gola*. iii. 57 (*vāsanā*). 5. *SiŚi, Gola*, iii. 53.

Āryabhaṭa II was undoubtedly aware of a formula for the surface of a sphere, though he has not expressly defined it anywhere. For he says, "the diameter of the earth is a little less than 2109 ; its circumference is 6625 ; and the area of its surface is 13971849."[1]

Now according to Āryabhaṭa II, $\pi = \frac{21600}{6876}$. Then

$$\text{Diameter of the earth} = \frac{6876}{21600}(\text{circumference of the earth})$$

$$= \frac{6876}{21600} \times 6625.$$

$$\text{Surface} = 6625\,(2109 - \tfrac{1}{24})$$

$$= 13971849 - \tfrac{1}{24}.$$

Thus it seems that Āryabhaṭa II employed the formula

Surface of a sphere = (circumference) × (diameter).

This formula is, however, expressly stated by Bhāskara II.[2] He further says :

"That (the area of a diametral section) multiplied by four is the net lying all over a round ball (i. e., the area of the surface of a sphere)."[3]

$$S = 4\pi r^2.$$

Bhāskara II has given a demonstration of this formula by means of the method of infinitesimals. We shall describe it later on.

Volume of a Sphere

Āryabhaṭa I writes :

"That (the area of a diametral section) multiplied by its own square-root is the exact volume of a sphere."[4]

That is to say, if r be the radius of a sphere, then according to Āryabhaṭa I,

$$\text{Volume of a sphere} = \pi r^2 \sqrt{\pi r^2}.$$

This formula is inaccurate. Śrīdhara says :

"Half the cube of the diameter of a sphere, then added with its eighteenth part, will give the volume."[5]

$$\text{Volume} = \frac{19 \times (\text{diameter})^3}{18 \times 2}.$$

1. *MSi*, xvi. 35-6. 2. *SiŚi*, *Gola*, iii. 52, 61. 3. *L*, p. 55.
4. *Ā*, ii. 7. 5. *TrіŚ*, R. 56.

Mahāvīra writes :

"Nine times the half of the cube of the semi-diameter is the *Vyāvahārika* volume of a sphere. Nine-tenth of that will be the very accurate volume."[1]

Āryabhaṭa II :

"The cube of the diameter of a sphere being halved and then added with its eighteenth part, will give its volume in cubic cubits : such is the formula taught (by the ancient teachers)."[2]

This formula was given before by Śrīdhara. It reappears also in the works of Śrīpati.[3] All the above-mentioned formulae for the volume of a sphere are more or less approximate. The truly accurate formula is, however, given by Bhāskara II. He says :

"That area of the surface multiplied by the diameter and divided by six, will be the accurate value of the volume of a sphere."[4]

That is to say, we shall have

$$\text{Volume} = \tfrac{1}{6}\,(\text{surface}) \times (\text{diameter}).$$

Now according to Bhāskara II,

$$\text{Surface} = 4\,(\text{area of a diametral section}),$$

$$\text{Area of a diametral section} = \tfrac{1}{4}\,(\text{circumference}) \times (\text{diameter}),$$

$$\text{Circumference} = \frac{22}{7}\,(\text{diameter}).$$

Therefore

$$\text{Volume} = \frac{22}{42}\,D^3,$$

$$= \left(1 + \frac{1}{21}\right)\frac{D^3}{2}.$$

Hence Bhāskara II writes :

"Half the cube of the diameter being added with its twenty-oneth part becomes the volume of a sphere."[5]

He has further observed that the volume of a sphere obtained by this formula is "rough" (*sthūla*). This is clearly so because that formula is derived with the rough value $\frac{22}{7}$ of π instead of its accurate value $\frac{3927}{1250}$.

1. *GSS*, viii. 28½.
2. *MSi*, xvi. 108.
3. *SiŚe*, xiii. 46.
4. *L*, p. 55.
5. *L*, p. 57.

Average Value

In measuring the volume of an excavation whose length, breadth or depth is different at different portions, the other two dimensions remaining the same, the Hindus take for all practical purposes the arithmetic mean of the varying elements. This mean value is technically called *sama-rajju* ("mean measure") by Brahmagupta, *samīkaraṇa* ("equalising value") by Mahāvīra, *sāmya* ("equability", i.e. "equivalent value") by Śrīpati and *samamiti* ("average value") by Bhāskara II.

Brahmagupta says :

"In an excavation having the same breadth at the face and bottom, the aggregates (of the partial products of lengths and depths) divided by the total (length) will be the mean measure (*sama-rajju*) of the depth."[1]

Example from Pṛthūdakasvāmi :

"A tank 30 cubits in length and 8 cubits in breadth contains within it five different excavations which subdivide the length into five portions of lengths four, five etc. (cubits). The depths (of these portions) are successively 9,7,7,3 and 2. Tell at once what is the mean depth of the excavation."

$$\text{Mean depth} = \frac{4.9+5.7+6.7+7.3+8.2}{4+5+6+7+8}$$
$$= \frac{150}{30}$$
$$= 5.$$

Therefore the volume of the tank

$$= 8.\ 30.\ 5 = 1200.$$

Mahāvīra writes :

"Find the half of the top and bottom dimensions ; the sum of all the halves divided by the number of them will be the equivalent value."[2]

"The sum of the depths (measured at different places) divided by the number of places will be the average depth."[3]

According to Bhāskara II,

"Calculate the breadth at several places : the sum of them divided by the number of places is the average value. Do in the same way in case of the length and depth."[4]

1. *BrSpSi*, xii. 44.
2. *GSS*, viii. 4.
3. *GSS*, viii. 23½.
4. *L*, p. 64.

11. Measurement of Heights and Distances

Shadow Reckoning

The *chāyā*, meaning literally "shadow", but implying truly the measurement by means of shadow of a gnomon, is a common topic for discussion in the Hindu treatises of mathematics. It is applied for measurement of time as well as of heights and distances. We shall, however, notice here only those rules which are related to its application in this latter aspect.[1]

Āryabhaṭa I says.

"Multiply the distance between the gnomon and the lamp-post[2] by the length of the gnomon and divide by the difference between the lengths of the gnomon and the lamp-post. The result will be the length of the shadow of the gnomon measured from its base."[3]

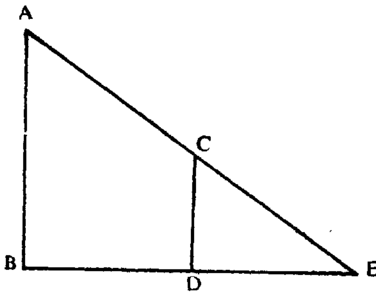


Fig. 20

AB = the lamp-post,
CD = the gnomon,
DE = the shadow of the gnomon.

$$DE = \frac{BD \times DC}{AB - CD}.$$

Similar rules are given by Brahmagupta[4], Mahāvīra[5], Srīpati[6] and Bhāskara II[7]. Some later writers[8] have described separately the formulae for calculating the height of the lamp from the length of the shadow and the distance of the gnomon, and the distance from the height of the lamp and the length of the shadow, though the same follow at once from the formula stated above.

Heights and Distances

Another and more useful problem is to find the height and distance of a far-off object. By way of illustration of the method employed a high light-post is generally taken into consideration. Then two gnomons of equal heights

1. The measurement of time by means of a gnomon is more fully treated in treatises on astronomy.
2. The Sanskrit original is *bhuja*. Ordinarily the term denotes a side of a triangle (or any rectilinear figure). All the commentators agree in interpreting it as implying here the lamp-post. Later rules are quite explicit. 3. *Ā*, ii. 15,
4. *BrSpSi*, xii, 53. 5. *GSS*, ix. 40½. 6. *SiSé*, xiii. 54. 7. *L*, p. 73.
8. See *GSS*, viii. 43, 45; *SiŚe*, xiii. 55; *L*, p. 74,

or the same gnomon successively, being set up in two places at a known distance apart, the two shadows are measured.

Āryabhaṭa I writes :

"The distance between the tips of the two shadows being multiplied by the length of a shadow and divided by the difference between the lengths of the two shadows gives the *koṭi*. That *koṭi* multiplied by the length of the gnomon and divided by the length of the shadow corresponding to it will be the height of the lamp-post."[1]

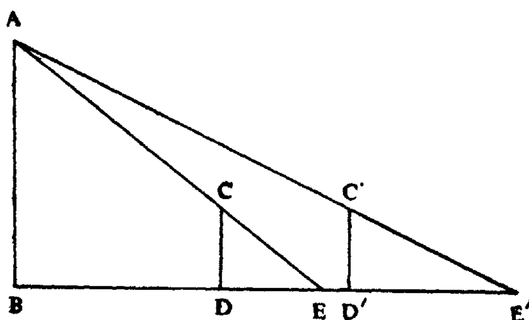


Fig. 21

AB is the lamp-post to be measured ; *CD*, *C' D'*=the gnomon in its two positions : and *DE*, *D' E'* = the shadows respectively. Then the rule says :

$$BE = \frac{EE' \times DE}{D'E' - DE}, \; BE' = \frac{EE' \times D'E'}{D'E' - DE},$$

$$AB = \frac{BE \times CD}{DE} = \frac{BE' \times CD}{D'E'}$$

These formulae are stated also by Brahmagupta[2] and Bhāskara II[3].

Brahmagupta's Rules.

The procedure to be adopted in actual practice in measuring the height of a distant object has been indicated by Brahmagupta as follows :

(i) Selecting a plane ground, the gnomon is fixed vertically in the position *CD*. Now the eye is put at the level of the ground at such a place *E* that *E*, *C* and *A* are in the same straight line. Then the distance *DE* of the eye from the gnomon is measured. It is called as *dṛṣṭi* (sight) by Brahmagupta. Similar observations are taken with the gnomon in a different position *C'D'* and the eye *E'*. The formulae to be applied then are the same as those stated above.

1. *Ā*. ii 16. 2. *BrSpSi*, xii. 54. 3. *L*, p. 75.

Brahmagupta redescribes them as follows :

"The displacement (of the eye) multiplied by a *dṛṣṭi* and divided by the difference of the two *dṛṣṭis* will give the distance of the base. The distance of the base multiplied by the length of the gnomon and divided by its own *dṛṣṭi* will give the height."[1]

(ii) Observations may also be taken, thinks Brahmagupta, by placing the gnomon horizontally on the level ground. In this case a graduated rod CR is fixed vertically at the extremity C of the gnomon CD nearer to the object to be measured. Then placing the eye at the other end D, the graduation P which is in a straight line with the tip of the object is noted. This gives the altitude CP. Brahmagupta calls it by the term *śalākā* (rod). Observations are taken again with the gnomon in the position $C'D'$.

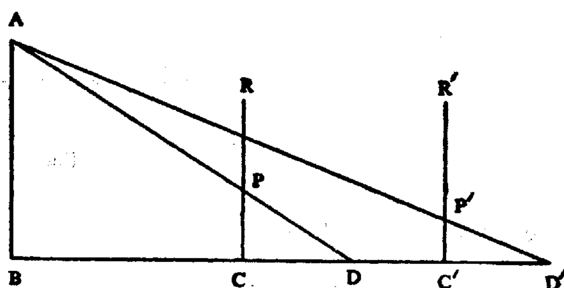


Fig. 22

Then Brahmagupta says :

"The displacement (of the gnomon) multiplied by the other *śalākā* and divided by the difference of the two *śalākās* will give the distance of the base. The distance of the base multiplied by the *śalākā* corresponding to it and divided by the length of the gnomon will give the height of the house etc."[2]

$$BD = \frac{DD' \times C'P'}{CP - C'P'}, \quad BD' = \frac{DD' \times CP}{CP - C'P'},$$

$$AB = \frac{BD \times CP}{CD} = \frac{BD' \times C'P'}{CD}.$$

(iii) Brahmagupta then gives a different method : Placing the eye at E, the gnomon is first directed towards the base B of the object and then

1. *BrSpSi*, xxii. 33. 2. *BrSpSi*, xxii, 32.

towards its tip A. From the front extremities G, G' of the gnomon in the two positions draw the perpendiculars GN, $G'N'$ to the ground. Also draw the perpendicular EM. Measure the distances MN, MN'.

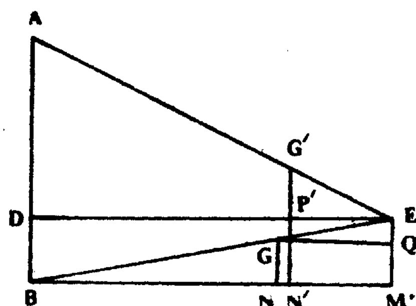


Fig. 23 (a)

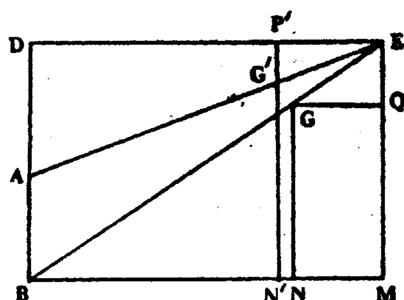


Fig. 23 (b)

Now it can be proved easily that

$$BM = \frac{ME \times MN}{ME - GN},$$

and

$$AB = ME + \frac{BM\,(G'N' - ME)}{MN'} \text{ in Fig 23 (a)}$$

or

$$= ME - \frac{BM\,(ME - G'N')}{MN'} \text{ in Fig. 23 (b)}$$

Hence Brahmagupta says :

"The distance between the feet of the altitudes (of the eye and the front extremity of the gnomon in the first observation) being divided by the difference between the altitudes and multiplied by the greater (altitude) gives the distance of the base. Multiply the distance of the base by the difference between the altitudes (of the eye and the front extremity of the gnomon in the second observation) and divide by the distance between the feet of these altitudes. Then subtract the quotient from the altitude of the eye, if the altitude of the front extremity of the gnomon (in the second observation) be less than the altitude of the eye ; or add, if greater. The result gives the height of the house. (Thus the height and distance of an

object can be determined) by means of observations of its base and tip."[1]

(iv) Another method of Brahmagupta is as follows: Placing the eye at E, at an altitude *ME* over the ground, then fix the gnomon *CD* in front in such a position that its lower end *D* will be in the line of sight of the bottom of the object *AB* and its upper end *C* in the line of sight of the top of the object. (Fig. 24). Also note the portion *DP'* of the gnomon below *EP*, the horizontal line of sight and the distance *EP'* of the eye from the gnomon. Then, says Brahmagupta:

"The distance of the eye from the gnomon multiplied by the altitude of the eye and divided by the portion of the gnomon below (the horizontal line of sight) will be the distance of the base. The distance of the base multiplied by the whole gnomon and divided by the distance of the eye from the gnomon will be the height."[2]

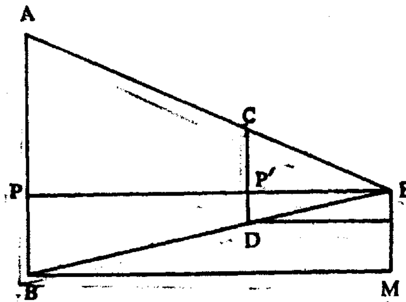


Fig. 24

$$BM = \frac{EP' \times ME}{DP'}, \qquad AB = \frac{BM \times CD}{EP'}$$

Bhāskara II.

For measuring the heights and distances of far-off objects, Bhāskara II gives two methods, one of which is taken from Brahmagupta. He remarks in general that observations should be made on a plane horizontal ground. Directing the gnomon towards the distant object perpendiculars are drawn from its two extremities on the plane of observation. The horizontal distance between them is the base (*bhuja*), the difference between them is the upright (*koṭi*) and the gnomon itself is the hypotenuse (*karṇa*) of the triangle of observation, says Bhāskara. (a) "Thus observing the bottom of the bamboo, multiply the base (of the triangle of observation) by the altitude of the eye and divide by the upright: the result is the horizontal distance between the self and the bamboo. Then observing the top of the bamboo, multiply the horizontal distance by the upright and divide by the base; the result

1. *BrSpSi*, xxii. 34-5. 2. *BrSpSi*, xxii. 36. For a fifth method see *BrSpSi*, xxii. 37.

added with the altitude of the eye is the height of the bamboo."[1]. (Fig. 24 (*a*) (*b*)) "Observe the top (of the bamboo) first in the standing posture and then again in the sitting posture. Divide each altitude by its base. The difference of the altitudes of the eye divided by the difference of those quotients gives the horizontal distance. The height of the bamboo can then be determined separately as before."[2]

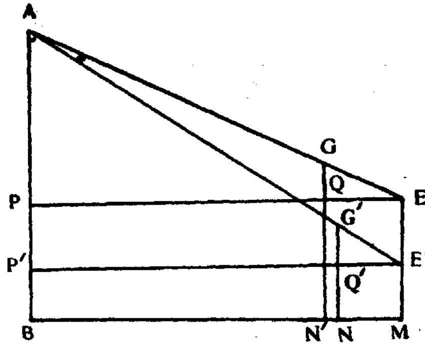


Fig. 25

$$PE = \frac{ME - ME'}{\frac{G'Q'}{E'Q'} - \frac{GQ}{EQ}},$$

$$AB = ME + \frac{PE \times GQ}{EQ} = ME' + \frac{PE \times G'Q'}{E'Q'}$$

1. *SiŚi*, *Golādhyāya*, *Yantrādhyāya*, 43-4.
2. *SiŚi*, *Golādhyāya*, *Yantrādhyāya*, 45-6.

ABBREVIATIONS

Ā = Āryabhaṭīya
ĀpŚl = Āpastamba Śulba
BrSpSi = Brāhma-sphuṭa-siddhānta
BŚl = Baudhāyana Śulba
GSS = Gaṇita-sāra-saṃgraha
KapS = Kapisthala Saṃhitā
KŚl = Kātyāyana Śulba
KṭS = Kāthaka Saṃhitā
L = Līlāvatī
MaiS = Maitrāyaṇīya Saṃhitā
MāŚl = Mānava Śulba
MSi = Mahā-siddhānta
ŚBr = Śatapatha Brāhmaṇa
ŚiDVṛ = Śiṣya-dhī-vṛddhida
SiŚe = Siddhānta-śekhara
SiŚi = Siddhānta-śiromaṇi
Triś = Triśatikā
TS = Tantra-saṃgraha

Indian Journal of History of Science, 18 (1) : 39—108 (1983)

HINDU TRIGONOMETRY

by

BIBHUTIBHUSAN DATTA AND AVADHESH NARAYAN SINGH

Revised by

KRIPA SHANKAR SHUKLA

Hussainganj Crossing, Behind Lakshman Bhawan
Lucknow

CONTENTS

1. TRIGONOMETRICAL FUNCTIONS. DEFINITIONS.

The Hindu name for the science of Trigonometry is *Jyotpatti-gaṇita* or "The science of calculation for the construction of the sine."[1] It is found as early as in the *Brāhma-sphuṭa-siddhānta* of Brahmagupta (628).[2] Sometimes that name is simplified into *Jyā-gaṇita* (or "The science of calculation of the sines").[3] In very recent years there has appeared the name *Trikoṇamiti*[4], which is a literal as well as phonetic rendering of the Greek name for the science.

The Hindus introduced and usually employed three trigonometrical functions, namely *jyā*, *koṭi-jyā* and *utkrama-jyā*. It should be noted that they are functions of an arc of a circle, but not of an angle. If AP be an arc of a circle with centre at O, then

[1]*Jyā* ("sine")+*utpatti* ("construction", "generating")+*gaṇita* ("the science of calculation").
[2]*BrSpSi*, xii. 66.
[3]Compare *SiTVi*, ii. 1.
[4]*Trikoṇa* ("triangle")+*miti* ("measure").

its *jyā*=PM, *koṭi-jyā*=OM and *utkrama-jyā*=$OA—OM=AM$. Hence their relation with modern trigonometrical functions will be

jyā $AP=R\sin\theta$, *koṭi-jyā* $AP=R\cos\theta$, *utkrama-jyā* $AP=R—R\cos\theta=R\text{versin}\,\theta$, where R is the radius of the circle and θ the angle subtended at the centre by the arc AP. Thus the values of the Hindu trigonometrical functions vary with the radius chosen. The earliest Hindu treatise in which the above trigonometrical functions are now found recorded is the *Sūrya-siddhānta*.

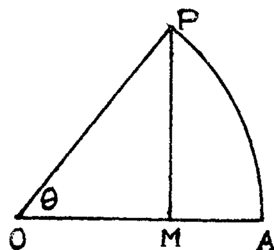


Fig. 1

Jyā

The Sanskrit word *jyā* means "a bow-string"; and hence "the chord of an arc", for the arc is called "a bow" (*dhanu, cāpa*). Its synonyms are *jīvā, siñjinī,*[1] *guṇa, maurvī,* etc. This trigonometrical function is also called *ardha-jyā*[2] ("half-chord") or *jyārdha*[3] ("chord-half"). Thus Bhāskara II (1150) explicitly observes, "It should be known that *ardha-jyā* is here called "*jyā*".[4] Parameśvara (1430) remarks:

"A part of a circle is of the form of a bow, so it is called the "bow" (*dhanu*). The straight line joining its two extremities is the "bow-string" (*jīvā*); it is really the "full-chord" (*samasta-jyā*). Half of it is here (called) the "half-chord" (*ardha-jyā*), and half that arc is called the "bow" of that half-chord. In fact the *R*sine (*jyā*) and *R*cosine (*koṭi-jyā*) of that bow are always half-chords."[5]

Kamalākara (1658) is more explicit. "Having seen the brevity", says he, "the half-chords are called *Jyā* by mathematicians in this (branch of) mathematics and are used accordingly."[6] The function *jyā* is sometimes distinguished as *krama-jyā*[7] or *kramārdha-jyā*,[8] from *krama*, "regular" or "direct" meaning "direct sine" or "direct half-chord".

It may be noted that the modern term *sine* is derived from the Hindu name. The Sanskrit term *jīvā* was adopted by the early Arab mathematicians but was pronounced as *jība*. It was subsequently corrupted in their tongue into *jaib*. The latter word was confused by the early Latin translators of the Arabic works such as Gherardo of Cremona (*c.* 1150 A.D.) with a pure Arabic word of alike phonetism but meaning differently "bosom" or "bay" and was rendered as *sinus*, which also signifies "bosom" or "bay".[9]

---

[1] *ŚiDVṛ*, ii. 9; *MSi*, iii. 2.
[2] *Ā*, i .10; *BrSpSi*, ii. 2.
[3] *SūSi*, ii. 15; *Ā*, ii. 11, 12; *BrSpSi*, xxi. 17, 22.
[4] *SiŚi, Graha*, iii. 2.
[5] *Ā*, ii. 11 (*Com.*).
[6] *SiTVi*, ii. 52.
[7] *PSi*, iv. 28 (*Kramaśo jyā*); *BrSpSi*, ii. 15; vii. 12.
[8] *ŚiDVṛ*, ii. 1.
[9] Cf. *Nouv. Ann. Math.*, XIII (1854), p. 393; Smith, *History* II, p. 616.

The degeneration and variations of the term *kramajyā* are still more interesting. In the Arabic tongue it was corrupted into *karaja* or *kardaja*. According to *Fihrist*, the title of a work of Ya'kūb ibn Ṭārik (*c.* 770 A.D.) is "On the table of *kardaja*." This table was copied from the *Brāhma-sphuṭa-siddhānta* of Brahmagupta. In the same connexion, al-Khowārizmī (825) used the variant *karaja*. In the Latin translations of the term we find several variants such as *kardaga, karkaya, gardaga* or *cardaga*. These terms had in foreign lands also the restricted uses for the arc of 3° 45′, sometimes of 15°.[1]

Koṭi-Jyā

The Sanskrit word *koṭi* means, amongst others "the curved end of a bow" or "the end or extremity in general"; hence in Trigonometry it came to denote "the complement of an arc to 90°."[2] So the radical significance of the term *koṭi-jyā* is "the *jyā* of the complementary arc". But it began early to be used as an independent technical term.[3] The modern term cosine appears to be connected with *koṭijyā*, for in Hindu works, particularly in the commentaries *koṭijyā* is often abbreviated into *kojyā*. When *jyā* became *sinus, kojyā* naturally became *ko-sinus* or *co-sinus*.

Utkrama-Jyā

Utkrama means "reversed", "going out" or "exceeding". Hence the term *utkrama-jyā* literally means "reversed sine". This function is so called in contradistinction to *krama-jyā*, for it is, rather its tabular values are, derived from the tabular values of the latter by subtracting the elements from the radius in the reversed order. Or in other words it is the exceeding portion of the *krama-jyā* taken into consideration in the reversed order. Thus it is stated:

"The (tabular) versed sines are obtained by subtracting from the radius the (tabular) sines in the reversed order."[4]

"They (*jyārdha*), (being subtracted from the radius), in the reversed order beginning from the end, will certainly give the versed sines, that is, the arrows."[5]

Again, it is noteworthy that from a table of differences of sines, the successive sines are obtained by adding the differences in the direct order (from the top) whereas the corresponding versed sines will be found by adding the elements in the reversed order (from the end). This fact has been particularly noted by Sūryadeva Yajvā (born 1191 A.D.) and Śrīpati (1039 A.D.). The former observes:

[1]Woepeke, F. "Sur le mot *kardaga* et sur une méthode indienne pour calcul les sinus", *Nouv. Ann. Math.*, XIII (1854), pp. 386-393; Braunmühl, A. *Geschichte der Trigonometrie*, 2 vols., Leipzig, 1900, 1903 (hereafter referred to as Braunmühl, *Geschichte*); Vol. I, pp. 44, 45, 78, 102, 110, 120; vide also Sarton's note on the point in *Isis*, xiv (1930), pp. 421f.

[2]In Hindu mathematics, the term *koṭi* also denotes "the side of a right-angled triangle."

[3]Compare *ŚiDVṛ*, ii. 30 (infra p. 10).

[4]*SūSi*, ii. 22.

[5]*BrSpSi*, xxi. 18; Compare also *MSi*, iii. 3.

"In order to get the direct sines (*krama-jyā*), these (tabular) differences of sines (*khaṇḍa-jyā*) should be added regularly from the beginning; and in order to determine the reversed sines (*utkrama-jyā*), they should be added in the reversed order from the end."[1]

Śrīpati says:

"The difference of sines are called *jyākhaṇḍa* (tabular "difference of sines"); (adding them) in the reversed way beginning from the end will be obtained the versed sines (*vyasta-jyā*) of the half-arcs equal to the 96th parts of the celestial circle."[2]

This function is also called *vyasta-jyā*[3] (from *vyasta*, "cast or thrown asunder", "reversed") or *viloma-jyā*[4] (from *viloma*, "reverse"). Occasionally it is termed *utkrama-jyārdha*.[5] Another name for it is "arrow" (*iṣu*, *bāṇa*).[6] Bhāskara II observes:

"What is really the arrow between the bow and the bowstring is known amongst the scholars here (i.e. in Trigonometry) as the versed sine."[7]

So also says Kamalākara (1658):

"What lies between the chord and the arc, like the arrow, is the versed sine."[8]

Tangent and Secant

The Hindus approached very near the tangent and secant functions and actually employed them in astronomical calculations, though they did not expressly recognise them as separate functions. The *Sūrya-siddhānta* gives the following rule for calculating the equinoctial midday shadow of the gnomon at a station:

"The sine of the latitude (of the station) multiplied by 12 and divided by the cosine of the latitude gives the equinoctial mid-day shadow."[9]

Here 12 is the usual height of a Hindu gnomon. So that

$$S=\frac{jy\bar{a}\ \phi\times h}{kojy\bar{a}\ \phi}$$

where ϕ denotes the latitude of the place, S=equinoctial mid-day shadow and h= gnomon. This is equivalent to

$$S=h\tan\theta.$$

[1] *Ā* i. 10 (*Com.*).
[2] *SiŚe*, xvi. 10,
[3] *BrSpSi*, ii. 5; *MSi*, iii. 3, 6.
[4] *ŚiDVṛ*, I, ii. 5.
[5] *SūSi*, ii. 22, 27.
[6] *BrSpSi*, xxi. 18.
[7] *SiŚi*, *Gola*, xiv. 5; Compare also *Graha*, ii. 20 (*Gloss*).
[8] *SiTVi*, ii. 58.
[9] *SūSi*, iii. 16.

Again to find the mid-day shadow (s) of the gnomon (h) and the hypotenuse (d), having known the meridian zenith distance (z) of the sun, we have the rules:[1]

$$s=h \tan z, \qquad d=h \sec z.$$

Similar rules occur in other astronomical works also.[2] In the *Gaṇita-sāra-saṃgraha* of Mahāvīra (850) by the term "shadow" of a gnomon is sometimes meant the ratio of the actual shadow to the height of the gnomon.[3] This ratio, as has been just stated, is equal to the tangent of the zenith distance of the sun.

Quadrants

A circle is ordinarily divided into four equal parts, called *vṛtta-pāda*, by two perpendicular lines, usually the east-to-west line and the north-to-south line. The quadrants are again classified into odd (*ayugma, viṣama*) and even (*yugma, sama*). Earlier Hindu writers do not explain this fact fully and particularly. Thus Bhāskara I (629) simply observes: "Three signs form a quadrant".[4] Lalla writes:

"Three anomalistic signs form a quadrant. The quadrants are successively distinguished as odd and even."[5]

But the description of Bhāskara II (1150) is very full. He says:

"Three signs together form a quadrant. In a circle there will be four such; and they should be successively called odd and even."[6]

He then explains it further thus:

"On a plane surface describe a circle of any specified radius with a pair of compasses. Mark on its circumference 360 degrees. Draw the east-to-west and north-to-south lines through its centre. These lines will divide the circle into quadrants, which should be taken into consideration in the leftwise manner (*savya-krama*, that is 'anti-clockwise'[7]) proceeding from the east-point (*prācī*); they should be called odd and even (quadrants) successively."[8]

Variation in value

As regards the variation in the value of a trigonometrical function as its argument changes, Bhāskara II observes as follows:

[1] *SūSi*, iii. 21.
[2] *PSi*, iv. 22.
[3] *GSS*, ix. 8½.
[4] *MBh*, iv. 1; *LBh*, ii. 1.
[5] *ŚiDVṛ*, ii. 10.
[6] *SiŚi*, *Graha*, ii. 19.
[7] The Sanskrit term *savya-krama* ordinarily signifies the "clockwise direction"; but it may also denote the "anti-clockwise direction."
[8] *SiŚi*, *Graha*, ii. 19 (*Gloss*).

"In the first quadrant, mark a point on the circumference of the circle at any optional distance from the east point. The perpendicular distance of that point from the east-to-west line is called the *R*sine (*doḥ-jyā*); and its distance from the north-to-south line is the *R*cosine (*koṭi-jyā*). The corresponding arcs are called *bhuja* and *koṭi*. (Starting from the east point) as the point gradually moves forward in the same way (i.e. anti-clockwise), the *R*sine increases and the *R*cosine decreases. When the point arrives at the end of the quadrant, the *R*cosine vanishes and the *R*sine is equal to the radius. Then in the second quadrant, the *R*cosine increases; at the end of that quadrant the *R*cosine is maximum (irrespective of sign) and the *R*sine vanishes."[1]

One fact perhaps deserves a particular notice here. It is that in Hindu trigonometry the *jyā* of an arc of 90° in a circle is equal to the radius of that circle. On account of that, the radius is called in Hindu mathematics by the terms *tri-jyā*, *tri-bha-jyā*, *tribhavana-jyā*, etc., every one of which literally means the "sine of three signs." The radius is also called *viṣkambhārdha*, *vyāsārdha*, or *ardha-vyāyāma* meaning the "semi-diameter". All these terms are very old.[2]

Functions of a complement or supplement

Sūrya-siddhānta says:

"In odd quadrants, the arc passed over gives the *R*sine, while the arc to be passed over gives the *R*cosine; and in the even quadrants, the arc to be passed over gives the *R*sine and that passed over gives the *R*cosine."[3]

Bhāskara I writes:

"In the odd quadrants the arc described and that to be described should respectively be known as the *bhuja* and *koṭi*; but in the even quadrants they are respectively the *koṭi* and *bhuja*; this is the fact."[4]

Lalla remarks:

"When (the anomaly[5] is) greater than 90°, it is subtracted from the semi-circle (i.e. 180°); when greater than the semi-circle, 180° is subtracted from it; when greater than 270°, it is subtracted from the complete circle (i.e. 360°); the remainder is called the (corresponding) *bhuja* by the expert in the subject.'[6]

[1]*Ibid*, ii. 20 (*Gloss*). The Sanskrit terms *jyā* and *kojyā* have been translated as *R*sine and *R*cosine because they are equal to $R \times$sine and $R \times$cosine respectively.

[2]Compare *ĀpŚlSū*, vii. 11 (*ardha-vyāyāma*); *Jambūdvīpasamāsa* of *Umāsvāti*, iv (*vyasārdha*); *Tattvārthādhigama-sūtra-bhāṣya*, iv. 14 (*viṣkambhārdha*).

[3]*SūSi*, ii. 30.

[4]*LBh*, ii. 1-2; Compare also *MBh*, iv. 8-9.

[5]It is in connection with the treatment of the anomaly that the remark of Lalla, as of several other Hindu mathematicians, occurs.

[6]*ŚiDVṛ*, ii. 10-11.

In the words of Brahmagupta:

"The *R*sine and *R*cosine (are obtained) in the odd quadrants from the arc passed over and to be passed over (respectively); and in the even quadrants in the reverse way."[1] Or,

"In the odd quadrants (the *R*sine is determined) from the arc described and in the even quadrants from the arc to be described."[2]

"(For the determination of) the *R*sine (proceed with the anomaly as it is) when the anomaly is less than three signs (i.e. 90°); when greater than three signs subtract it from six signs; when greater than six signs, subtract six signs (from it); when greater than nine signs, subtract it from the complete circle."[3]

Mañjula (932) says:

"In the odd quadrants, the *bhuja* and *koṭi* are (to be calculated) from the arc described and that to be described (respectively); but in the even quadrants in the contrary way."[4]

His commentator and younger contemporary Praśastidhara (962) dilates upon this point thus:

"In the odd quadrant, where the anomaly is less than three signs (i.e. 90°), the *R*sine should be calculated from it and the *R*cosine should be calculated after subtracting that from 90°. In the even quadrant, where the anomaly exceeds 90° but is less than 180°; in that case the *R*sine should be taken after subtracting it from 180° and the cosine after subtracting 90° from it. In the odd quadrant, where the anomaly is greater than 180°, but less than 270°, the *R*sine should be calculated after subtracting 180° from it and the *R*cosine after subtracting it from 270°. In the even quadrant when the anomaly exceeds 270°, but is less than 360°, the *R*sine is determined after subtracting it from 360°, and the *R*cosine after subtracting 270° from it."[5]

Śrīpati (*c.* 1039) remarks:

"In the odd and even quadrants, the arc passed over and to be passed over (respectively) is the *bhuja* and the *koṭi* is otherwise. Or, as the learned have said, the *R*sine of 90° minus the anomaly is the *R*cosine (of the anomaly)."[6]

---

[1]*BrSpSi*, ii. 12.
[2]*KK, I, i.* 16.
[3]*KK. I, i.* 16.
[4]*LMā*, ii. 2.
[5]Commentary on the same.
[6]*SiŚe*, iii. 13.

And Bhāskara II:

"In the odd quadrants, the arc passed over and in the even quadrants the arc to be passed is the *bhuja*. Ninety degrees minus the *bhuja* is said to be the *koṭi*."[1]

The above results can be represented graphically thus:

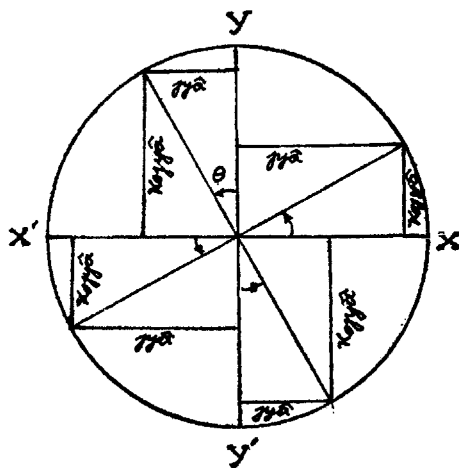


Fig. 2

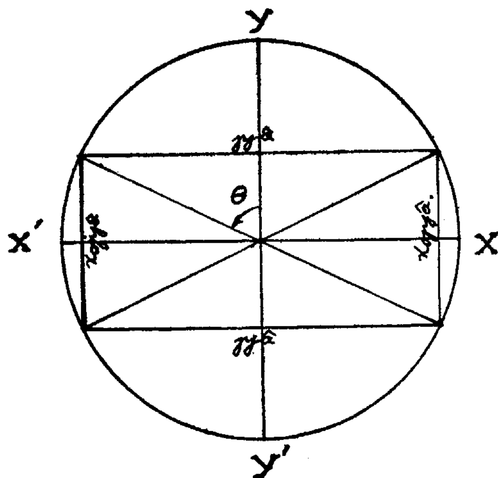


Fig. 3

Relation between Functions

Varāhamihira says:

"The *R*sine of 90° minus latitude is the *R*cosine of the latitude."[2]

Lalla:

"The square of the base-sine (*bhuja-jyā*) is subtracted from the square of the radius; the square root of the remainder is the *R*cosine; or it is the *R*sine of 90° minus the *bhuja* arc."[3]

$$\sqrt{R^2 - (jy\bar{a}\,\alpha)^2} = kojy\bar{a}\,\alpha$$

or

$$kojy\hat{a}\,\alpha = jy\hat{a}\,(90^\circ - \alpha)$$

where *kojyā* is the usual Hindu symbol for *koṭi-jyā*.

[1] *SiŚi, Graha*, ii. 19.
[2] *PSi*, iv. 28.
[3] *ŚiDVṛ*, ii. 30.

Brahmagupta says:

"The radius diminished by the versed *R*sine of an arc or of its complement will give the *R*sine of the other. The square-root of the difference of the square of the radius and that of the *R*sine of an arc or of its complement will be the *R*sine of the other."[1]

$$R - utjy\bar{a}\,\alpha = jy\bar{a}\,(90^\circ - \alpha),$$
$$R - utjy\bar{a}\,(90^\circ - \alpha) = jy\bar{a}\,\alpha,$$
$$\sqrt{R^2 - (jy\bar{a}\,\alpha)^2} = jy\bar{a}\,(90^\circ - \alpha),$$
$$\sqrt{R^2 - \{jy\bar{a}(90^\circ - \alpha)\}^2} = jy\bar{a}\,\alpha$$

where *utjyā* is the usual abbreviation for *utkrama-jyā*.

"The direct *R*sine of the excess of an arc over 90° added to the radius will give versed *R*sine of that arc."[2]

$$R + jy\bar{a}\,(\alpha - 90^\circ) = utjy\bar{a}\,\alpha,$$

where $\alpha > 90^\circ$.

Śrīpati writes:

"The square of the radius is diminished by the square of the *R*sine; the square-root of the remainder will be the *R*cosine. Again the square-root of the square of the radius minus the square of the *R*cosine will be the *R*sine. The radius minus the versed *R*sine of the complement of an arc is equal to the *R*sine of the arc, and minus the versed *R*sine of the arc becomes the *R*sine of the other (i.e. complement)."[3]

The treatment of Bhāskara II is exhaustive. He says:

"Subtract from the radius the direct *R*sine of an arc and of its complement; the results will be the versed *R* sines of the complement and the arc (respectively). Subtract from the radius the versed *R*sine of an arc and of its complement; the remainders will be the direct *R*sines of the complement and the arc (respectively)."[4]

$$R - jy\bar{a}\,\alpha = utjy\bar{a}\,(90^\circ - \alpha),$$
$$R - jy\bar{a}\,(90^\circ - \alpha) = utjy\bar{a}\,\alpha,$$
$$R - utjy\bar{a}\,\alpha = jy\bar{a}\,(90^\circ - \alpha),$$
$$R - utjy\bar{a}\,(90^\circ - \alpha) = jy\bar{a}\,\alpha.$$

[1] *BrSpSi*, xiv. 7.
[2] *Ibid*, vii. 12.
[3] *SiŚe*, iii. 14.
[4] *SiŚi*, *Graha*, ii. 20; also *Gola*, v. 2; xiv. 5.

"The square of the *R*sine of an arc and of its complement are (severally) subtracted from the square of the radius, the square-roots of the results are (respectively) the *R*sines of the complement and of the arc."[1]

$$\sqrt{R^2-(jy\bar{a}\ \alpha)^2}=jy\bar{a}\ (90^\circ-\alpha);\quad \sqrt{R^2-\{jy\bar{a}\ (90^\circ-\alpha)\}^2}=jy\bar{a}\ \alpha.$$

"The square of the radius is diminished by the square of the *R*sine of an arc; the square-root of the result is the *R*cosine of the arc."[2]

$$\sqrt{R^2-(jy\bar{a}\ \alpha)^2}=kojy\bar{a}\ \alpha.$$

Kamalākara writes:

"The square-root of the square of the radius diminished by the square of the *R*sine of an arc, is the *R*cosine of the arc; similarly, the square-root of the square of the radius diminished by the *R*cosine of an arc, is the *R*sine of the arc. Again, the *R*sines of an arc and its complement when subtracted from the radius will give the versed *R*sines of the complement and the arc (respectively)."[3]

$$\sqrt{R^2-(jy\bar{a}\ \alpha)^2}=kojy\bar{a}\ \alpha,\quad \sqrt{R^2-(kojy\bar{a}\ \alpha)^2}=jy\bar{a}\ \alpha,$$
$$R-jy\bar{a}\ \alpha=utjy\bar{a}\ (90^\circ-\alpha),\quad R-jy\bar{a}\ (90^\circ-\alpha)=utjy\bar{a}\ \alpha.$$

Change of Sign of a Function

The Hindus were fully aware of the changes of sign of a trigonometrical function according as its argument lies in different quadrants. Though nowhere do we find any systematic treatment of this principle in any Hindu work there are ample concrete instances of its application in almost all their important astronomical treatises. Thus it is stated in the *Sūrya-siddhānta:*

"The *śīghra koṭiphala* is positive, when the *kendra* (mean anomaly) lies in a position beginning with the Capricorn; and it is to be subtracted from the radius in a position beginning with the Cancer."[4]

Now according to the *Sūrya-siddhānta* and other Hindu astronomical works, the *śīghra koṭiphala* (the result derived from the complement of the distance from the conjunction) is given by $D\cos\theta$, where θ is the *śīghra kendra*[5] (the distance of the mean planet from its apex of swiftest motion; hence mean *śīghra* anomaly) and *D*, a certain known constant. The Cancer is the fourth sign of the Zodiac and Capricorn is the tenth sign. Again the motion of the mean planet is anti-clockwise. Hence it is clear from

[1] *SiŚi, Graha*, ii. 21.
[2] *SiŚi, Gola*, v. 2; xiv. 4.
[3] *SiTVi*, ii. 56-7.
[4] *SūSi*, ii. 40.
[5] "Subtract the longitude of a planet from that of its apex of slowest motion (*mandocca*); so also subtract it from that of its apex of swiftest motion (conjunction); the result (in either case) is its *Kendra*." *SūSi*, ii. 29.

the above rule that the author was aware that the cosine of an angle lying between 0° and 90° or between 270° and 360° is positive and that it is negative when the angle lies between 90° and 270°.

Again it has been said:

"In case of the *manda* and *śīghra* corrections of all planets, the *phala* (equation) will be positive, if the *kendra* lies in the six signs beginning with the Aries and it will be negative in the six signs beginning with the Libra."[1]

Now the *phala* is defined as arc ($D' \sin \theta$), where D' does not change sign. Hence clearly the author knows that the sign is positive in the first two quadrants and negative in the other two quadrants.

Similar rules are found in other treatises of astronomy.[2] The statement of Mañjula (932) is more explicit and fuller. He says:

"The (mean) planet when diminished by its apogee or aphelion is the *kendra* (mean anomaly). Its *R*sine is positive or negative in the upper or lower halves (of the quadrants); and its *R*cosine is positive, negative, negative, and positive (respectively) according to the (successive) quadrants."[3]

Thus the Hindus knew very early what in modern trigonometrical notations will be expressed as—

$$\sin(\pi \mp \theta) = \pm \sin \theta, \qquad \cos(\pi \mp \theta) = -\cos \theta$$

$$\sin(2\pi - \theta) = -\sin \theta, \qquad \cos(2\pi - \theta) = +\cos \theta$$

$$\sin\left(\frac{\pi}{2} \mp \theta\right) = +\cos \theta, \qquad \cos\left(\frac{\pi}{2} \mp \theta\right) = \pm \sin \theta$$

$$\sin\left(\frac{3\pi}{2} \mp \theta\right) = -\cos \theta, \qquad \cos\left(\frac{3\pi}{2} \mp \theta\right) = \mp \sin \theta$$

Again it has been stated before that according to a rule of Brahmagupta

$$R + jy\bar{a}\,(\alpha - 90^\circ) = utjy\bar{a}\ \alpha.$$

But by definition,

$$utjy\bar{a}\ \alpha = R - kojy\bar{a}\ \alpha = R - jy\bar{a}\,(90^\circ - \alpha).$$

[1] *SūSi*, ii. 45.
[2] For instance *Ā*, iii. 22; *MBh*, iv. 5, 9; *LBh*, ii. 6; *ŚiDVṛ*, ii. 32; *BrSpSi*, ii. 14ff.
[3] *LMā*, ii. 1.

These clearly show that the author knows that the value of the sine function changes sign along with its argument.

Or symbolically

$$\sin(\pm\theta) = \pm\sin\theta.$$

2. Trigonometrical Formulae

(1) $\sin^2\theta + \cos^2\theta = 1$

It has been stated before that according to Hindu astronomers, if α be an arc of a circle of radius R

$$\sqrt{R^2 - (jy\bar{a}\ \alpha)^2} = kojy\bar{a}\ \alpha,\quad \sqrt{R^2 - (kojy\bar{a}\ \alpha)^2} = jy\bar{a}\ \alpha.$$

These are of course equivalent to the modern formulae

$$\sqrt{1-\sin^2\theta} = \cos\theta,\quad \sqrt{1-\cos^2\theta} = \sin\theta$$

or $\sin^2\theta + \cos^2\theta = 1$,

where θ is the angle subtended at the centre of the circle by the arc α.

(2) $4\sin^2\frac{\theta}{2} = \sin^2\theta + \text{versin}^2\,\theta$

This formula has been stated first by Varāhamihira (505).

He says:

"To find the *R*sine of any other desired arc, double the arc and subtract from the quarter of a circle; diminish the radius by the *R*sine of the remainder. The square of half the result is added to the square of half the *R*sine of the double arc. The square-root of the sum is the desired *R*sine."[1]

Let the arc XP = arc $PQ = \alpha$; then arc $QY = 90° - 2\alpha$.

Now

$$XQ^2 = QT^2 + TX^2$$
$$\text{or } 4\,XD^2 = QT^2 + TX^2$$
$$\text{or } XD^2 = (QT/2)^2 + (TX/2)^2.$$

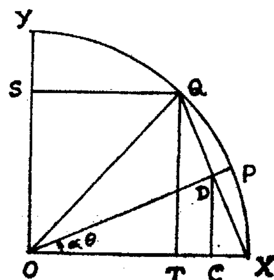


Fig. 4

[1] *PSi*, iv. 2f.

Hence

$$(jy\bar{a}\,\alpha)^2 = \left(\frac{jy\bar{a}\,2\alpha}{2}\right)^2 + \left(\frac{R - jy\bar{a}\,(90^\circ - 2\alpha)}{2}\right)^2;$$

which is equivalent to

$$4 \sin^2 \theta = \sin^2 2\theta + \text{versin}^2\, 2\theta.$$

Āryabhaṭa I (499) seems to have been aware of this formula before Varāhamihira. It reappears also in later works.

Brahmagupta says:

"The sum of the squares of the *R*sine and versed *R*sine of the same arc is divided by four; subtract this quotient from the square of the radius. Take the square-root of the two results. The former will be the *R*sine of half that arc, and the other the *R*sine of the arc equal to the quarter circle less that half."[1]

The formula has been described almost similarly by Śrīpati (1039)[2]. Bhāskara II (1150) writes very briefly thus:

"Half the square-root of the sum of the square of the *R*sine and of the versed *R*sine of an arc, will be the *R*sine of half that arc."[3]

Parameśvara (1430) says:

"The square-root of the sum of the square of the *R*sine and of the versed *R*sine of an arc is the 'whole chord' (*samasta-jyā*) of that arc. Half that is the half-chord (i.e. the *R*sine) of half that arc."[4]

(3) $2 \sin^2 \frac{\theta}{2} = 1 - \cos \theta$

This is given first by Varāhamihira. He says:

"Twice any desired arc is subtracted from three signs (i.e. 90°), the *R*sine of the remainder is subtracted from the *R*sine of three signs. The result multiplied by sixty is the square of the *R*sine of that arc."[5]

---

[1] *BrSpSi*, xxi. 20f.
[2] *SiŚe*, xvi. 14-5.
[3] *SiŚi*, *Gola*, v. 4; xiv. 10.
[4] Quoted in his commentary of *Ā*, ii. 11.
[5] *PSi*, iv. 5.

In the fig. 4, page 50, since the triangles XCD and XDO are similar, we have:

$$XD : XC :: XO : XD$$
$$\therefore XD^2 = XO.XC = \tfrac{1}{2} XO.XT$$

Hence $(jy\bar{a}\ \alpha)^2 = \frac{1}{2} R \left\{R - jy\bar{a}\ (90^\circ - 2\alpha)\right\}$.

The factor $\frac{1}{2} R$ on the right-hand side has been stated by Varāhamihira as 60 since he has taken the value of the radius to be equal to 120. In modern notations, the above formula becomes—

$$\sin^2 \theta = \tfrac{1}{2} (1 - \cos 2\theta).$$

This also follows easily from the preceding formula.

Brahmagupta says:

"The square-root of the fourth part of the versed *R*sine of an arc multiplied by the diameter is the *R*sine of half that arc."[1]

Bhāskara II writes:

"Or, the square-root of half the product of the radius and the versed *R*sine of an arc, will be the *R*sine of half that arc."[2]

He has further given the following proof of it.[3]

Since $kojy\bar{a}\ \alpha = R - utjy\bar{a}\ \alpha$

so that squaring $(kojy\bar{a}\ \alpha)^2 = R^2 + (utjy\bar{a}\ \alpha)^2 - 2R.\ utjy\bar{a}\ \alpha$

Therefore $R^2 - (kojy\bar{a}\ \alpha)^2 = 2R.\ utjy\bar{a}\ \alpha - (utjy\bar{a}\ \alpha)^2$

Or, $(jy\bar{a}\ \alpha)^2 = 2R.\ utjy\bar{a}\ \alpha - (utjy\bar{a}\ \alpha)^2$

$(jy\bar{a}\ \alpha)^2 + (utjy\bar{a}\ \alpha)^2 = 2R.\ utjy\bar{a}\ \alpha$

But by the formula (2), the righthand side is equal to

$$4\left(jy\bar{a}\ \frac{\alpha}{2}\right)^2.$$

Hence, $jy\bar{a}\ \frac{\alpha}{2} = \sqrt{\tfrac{1}{2} R.\ utjy\bar{a}}$.

This rule of Bhāskara II together with his proof has been reproduced by Kamalākara.[4]

[1] *BrSpSi*, xxi. 23.
[2] *SiŚi, Gola*, v. 5; xiv. 10.
[3] *SiŚi, Gola* (Gloss).
[4] *SiTVi*, ii. 78 and its commentary.

(4) $\sin \frac{1}{2}(90°\pm\theta) = \sqrt{\frac{1}{2}(1\pm \sin\theta)}$

This formula first appears in the works of Āryabhaṭa II (950). He says:

"The *R*sine of any arc multiplied by the radius is subtracted from or added to the square of the maximum value of the *R*sine; the square-root of half the results are extracted. These will be the *R*sine of 45° decreased or increased by half that arc."[1]

Let the arc XP be denoted by α. Bisect the complementary arc YP at Q. Then

$$\begin{aligned} YP^2 &= YN^2+NP^2 \\ &= (OY-PM)^2+PN^2 \\ &= OY^2+PM^2+OM^2-2\,OY.PM. \end{aligned}$$

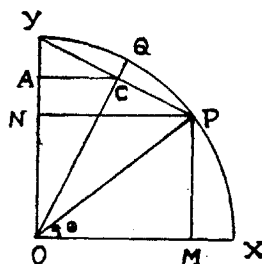


Fig. 5

Therefore,

$$4\,PC^2=2\,(OP^2-OP.PM).$$

Hence, $jy\bar{a}\,\frac{1}{2}(90°-\alpha) = \sqrt{\frac{1}{2}(R^2 - R.\,jy\bar{a}\,\alpha)}$.

Similarly it can be proved that

$$jy\bar{a}\,\frac{1}{2}(90°+\alpha) = \sqrt{\frac{1}{2}(R^2+R.\,jy\bar{a}\,\alpha)}.$$

These are of course equivalent to

$$\sin \frac{1}{2}(90° \pm \theta) = \sqrt{\frac{1}{2}(1 \pm \sin\theta)}.$$

Bhāskara II (1150) writes:

"The square of the radius is diminished or increased by the product of the radius and the *R*sine of an arc; the square-root of half the results will be the *R*sine of the half of 90° minus or plus that arc."[2]

Kamalākara defines:

"The product of the radius and the *R*sine of an arc is added to or subtracted from the square of the radius. The square-root of the half of the results are taken. They will respectively be the *R*sine of the half of three signs plus or minus the arc."[3]

[1] *MSi*, iii. 2.
[2] *SiŚi, Gola*, xiv. 12.
[3] *SiTVi*, ii. 93

He adduces the following proof of it:[1]

$R \pm jy\bar{a}\ \alpha = utjy\bar{a}\ (90^\circ \pm \alpha)$

Squaring and adding $\{jy\bar{a}\ (90^\circ \pm a)\}^2$ to both the sides, we get
$R^2 + (jy\bar{a}\ \alpha)^2 + \{jy\bar{a}\ (90^\circ \pm \alpha)\}^2 \pm 2R\ jy\bar{a}\ \alpha = \{jy\bar{a}\ (90^\circ \pm \alpha)\}^2 + \{(utjy\bar{a}\ (90^\circ \pm \alpha)\}^2$
or, $2\ (R^2 \pm R\ jy\bar{a}\ \alpha) = 4\ \{jy\bar{a}\ \frac{1}{2}\ (90^\circ \pm \alpha)\}^2$, by formulae (1) and (2).

(5) $2\ \cos^2 \frac{\theta}{2} = 1 + \cos\theta$

Bhāskara II remarks that if the arc α in the formula

$jy\bar{a}\ \frac{1}{2}\ (90^\circ \pm \alpha) = \sqrt{\frac{1}{2}(R^2 \pm R\ jy\bar{a}\ (90^\circ - \alpha)}$

be substituted by its complement $90^\circ - \alpha$, it will still be true.[2]

So that, $jy\bar{a}\ \frac{1}{2}\ (90^\circ \pm \overline{90^\circ - \alpha}) = \sqrt{\frac{1}{2}\{R^2 \pm R\ jya(90^\circ - \alpha)\}}$

which leads to, $2\cos^2 \frac{\theta}{2} = 1 + \cos\theta$, $2\sin^2 \frac{\theta}{2} = 1 - \cos\theta$.

Kamalākara says:

"Half the *R*cosine of an arc is added to the *R*sine of one sign (i.e. 30°) and the sum is multiplied by the radius; the square-root of the product should be known by the intelligent as the *R*cosine of half that arc."[3]

$kojy\bar{a}\frac{\alpha}{2} = \sqrt{R\ (jy\bar{a}\ 30^\circ + \frac{1}{2}\ kojy\bar{a}\ \alpha)}$

or $\cos^2 \frac{\theta}{2} = \sin 30^\circ + \frac{1}{2}\cos\theta = \frac{1}{2}(1 + \cos\theta)$

(6) $\sin^2 (45^\circ - \theta) = \frac{1}{2}\ (\cos\theta - \sin\theta)^2$

Bhāskara II says:

"The square of the difference of the *R*sine and *R*cosine of an arc is halved; the square-root of the result is equal to the *R*sine of half the difference between that arc and its complement."[4]

---

[1] *SiTVi*, (Gloss).
[2] *SiŚi*, *Gola*, xiv. 12.
[3] *SiTVi*, ii. 91.
[4] *SiŚi*, *Gola*, xiv. 14.

Denote the arc *XP* by α ; cut off the arc *YQ* equal to the arc *XP*. Bisect the chord *PQ* by the point *D*.

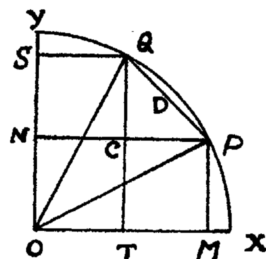


Fig. 6

Then, $CP = PN—CN=PN—QS$
$= kojyā\ \alpha — jyā\ \alpha$
$= QT—PM=CQ.$

Therefore, $PQ^2 = 2\ CP^2$
$PD^2 = \frac{1}{2}\ CP^2$

or, $jyā\ \frac{1}{2}\ (\overline{90°—\alpha}—\alpha) = \sqrt{\frac{1}{2}(kojyā\ \alpha\ —jyā\ \alpha)^2}$

which is equivalent to
$\sin\ (45°—\theta) = \sqrt{\frac{1}{2}\ (\cos\theta—\sin\theta)^2}.$

Kamalākara writes :

"The *R*sine of half the difference between an arc and its complement should be known by the intelligent in this (science) as equal to the square-root of half the square of the difference of the *R*sine of the arc and of its complement."[1]

His proof of the formula is substantially the same as that stated above.

(7) $\cos 2\ \theta = 1—2\ \sin^2\theta$

Bhāskara II gives:

"The square of the *R*sine of an arc is divided by half the radius; the difference between this quotient and the radius is equal to the *R*sine of the difference between that arc and its complement."[2]

In the fig. 4 on page 50

$QX^2 = QT^2+TX^2=QT^2+(OX—OT)^2$
$= QT^2+OT^2+OX^2—2\ OX.OT.$
or $4\ XD^2 = 2\ OX^2—2\ OX.QS.$

Hence, $QS=OX—\dfrac{XD^2}{OX/2}$

So that $jyā(90°—2\alpha) = R—\dfrac{(jyā\alpha)^2}{R/2}$

[1] *SiTVi*, ii. 95.
[2] *SiŚi*, *Gola*, xiv. 15.

which is the same as

$$\cos 2\theta = 1 - 2\sin^2\theta.$$

This formula is practically the same as (3). In the words of Kamalākara:

"Twice the square of the *R*sine of an arc is divided by the radius, the quotient is subtracted from the radius; the remainder will be the *R*sine of the difference of the arc and its complement."[1]

(8) $\sin^2\theta + \text{versin}^2\theta = 2 \text{ versin } \theta$

(9) $2 \sin\theta \cos\theta + [\text{versin } \theta - \text{versin } (90^\circ - \theta)]^2 = 1$

(10) $(1 + \sin\theta).\ \text{versin } (90^\circ - \theta) = \cos^2\theta$

(11) $2 \sin\theta \pm [\text{versin } \theta \sim \text{versin } (90^\circ - \theta)]$

$$= \sqrt{2 - [\text{versin } \theta \sim \text{versin } (90^\circ - \theta)]^2},$$

according as $\sin\theta \lesseqgtr \cos\theta$

(12) $(\cos\theta + \sin\theta)^2 + [\text{versin } \theta \sim \text{versin } (90^\circ - \theta)]^2 = 2.$

Formulae (8) to (12) and similar others occur in the *Vaṭeśvara-siddhānta* of Vaṭeśvara (904).

3. Addition and Subtraction Theorems

Bhāskara II (1150) says:

"The *R*sines of any two arcs of a circle are reciprocally multiplied by their *R*cosines; the products are then divided by the radius; the sum of the quotients is equal to the *R*sine of the sum of the two arcs; and their difference is the *R*sine of the difference of the arcs."[2]

If α and β be any two arcs, then the rule says:

$$jy\bar{a}\ (\alpha \pm \beta) = \frac{jy\bar{a}\ \alpha .\ kojy\bar{a}\ \beta}{R} \pm \frac{kojy\bar{a}\ \alpha .\ jy\bar{a}\ \beta}{R}$$

which is equivalent to

$$\sin(\theta \pm \phi) = \sin\theta \cos\phi \pm \cos\theta \sin\phi.$$

---

[1] *SiTVi*, ii. 96.
[2] *SiŚi, Gola*, xiv. 21f.

In the words of Kamalākara (1658):

"The quotients of the *R*sines of any two arcs of a circle divided by its radius are reciprocally multiplied by their *R*cosines; the sum and difference of them (the products) are equal to the *R*sine of the sum and difference respectively of the two arcs."[1]

The rule for finding the *R*cosine of the sum and difference of two arcs of a circle is enunciated by Kamalākara thus:

"The product of the *R*cosines and of the *R*sines of two arcs of a circle are divided by its radius; the difference and sum of them (the quotients) are equal to the *R*cosine of the sum and difference (respectively) of the two arcs."[2]

$$\textit{kojyā}\ (\alpha \pm \beta) = \frac{\textit{kojyā}\ \alpha .\ \textit{kojyā}\ \beta}{R} \mp \frac{\textit{jyā}\ \alpha .\ \textit{jyā}\ \beta}{R}$$

which is equivalent to

$$\cos(\theta \pm \phi) = \cos\theta \cos\phi \pm \sin\theta \sin\phi.$$

Though we do not find this *R*cosine theorem in the printed editions of the works of Bhāskara II, we are quite sure that it was known to him. For it has been attributed to him by his most relentless critic Kamalākara[3] as well as by his commentator Muniśvara.

The above theorems can be proved by methods algebraical as well as geometrical. Several such proofs were given by previous writers, observes Kamalākara[4] (1658). Unfortunately we have not been able to trace them as yet. The following two geometrical proofs are found in the *Siddhānta-tattva-viveka*[5] of Kamalākara.

First Proof. Let the arc $YP = \beta$, and arc $YQ = \alpha$; α being greater then β. Join OP, OQ.

Draw PN, PM perpendicular to OY, OX respectively. Also draw QS perpendicular to OY, and produce it to meet the circle again at Q'. Draw QT, $Q'T'$ perpendicular to OP. Then $PN = \textit{jyā}\ \beta$, $ON = \textit{kojyā}\ \beta$, $QS = \textit{jyā}\ \alpha$, $OS = \textit{kojyā}\ \alpha$; $PG = \textit{kojyā}\ \beta - \textit{kojyā}\ \alpha$, $QG = \textit{jyā}\ \alpha + \textit{jyā}\ \beta$, $QT = \textit{jyā}\ (\alpha + \beta)$, $PT = R - \textit{kojyā}\ (\alpha + \beta)$.

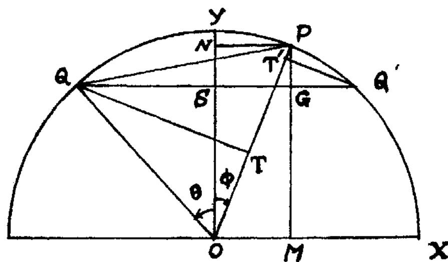


Fig. 7

[1] *SiTVi*, ii. 68.
[2] *Ibid*, ii. 69.
[3] Kamalākara remarks:
"*Evamānayanaṁ cakre pūrvaṁ svīyaśiromaṇau,*
Bhāvanābhyāmatispaṣṭaṁ saṁyagāryo' pi Bhāskaraḥ"
—*SiTVi*, ii. 70.
Or "This theorem, which is evident from the two *Bhāvanās*, was stated before also by the highly respected Bhāskara in his (*Siddhānta-*) *śiromaṇi*."
[4] "*Tasya cānayanasyāryaiḥ siddhāntajñaiḥ puroditā,*
Vāsanā bahubhiḥ svasvabuddhivaicitryataḥ sphuṭāḥ"—*SiTVi*, ii. 71.
Or "Many correct proofs of this theorem were given before by the learned authors of the *Siddhāntas* according to the manifoldness of their intelligence."
[5] ii. 68-9 (Gloss).

Now $PG^2+QG^2=QP^2=QT^2+PT^2$.

Therefore, substituting the values

$$(kojy\bar{a}\ \beta-kojy\bar{a}\ \alpha)^2+(jy\bar{a}\ \alpha+jy\bar{a}\ \beta)^2=\{jy\bar{a}\ (\alpha+\beta))\}^2+\{R-kojy\bar{a}\ (\alpha+\beta)\}^2$$

Simplifying we get

$$kojy\bar{a}\ (\alpha+\beta)=\frac{1}{R}(kojy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta-jy\bar{a}\ \alpha.\ jy\bar{a}\ \beta)$$

which is equivalent to

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi.$$

Again

$$R^2-\{kojy\bar{a}\ (\alpha+\beta)\}^2=\frac{1}{R^2}\{R^4-(kojy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta-jy\bar{a}\ \alpha.\ jy\bar{a}\ \beta)^2\}$$

$$=\frac{1}{R^2}[\{(jy\bar{a}\ \alpha)^2+(kojy\bar{a}\ \alpha)^2\}\times$$

$$\{(jy\bar{a}\ \beta)^2+(kojy\bar{a}\ \beta)^2\}-(kojy\bar{a}\ \alpha.kojy\bar{a}\ \beta-jy\bar{a}\ \alpha.jy\bar{a}\ \beta)^2]$$

or, $jy\bar{a}\ (\alpha+\beta)=\frac{1}{R}(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta+kojy\bar{a}\ \alpha.\ jy\bar{a}\ \beta)$,

which is

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi.$$

Since

$$PG^2+Q'G^2=Q'P^2=Q'T'^2+PT'^2,$$

we have

$$(kojy\bar{a}\ \beta-kojy\bar{a}\ \alpha)^2+(jy\bar{a}\ \alpha-jy\bar{a}\ \beta)^2=\{jy\bar{a}\ (\alpha-\beta)\}^2+\{R-kojy\bar{a}\ (\alpha-\beta)\}^2.$$

Therefore,

$$kojy\bar{a}\ (\alpha-\beta)=\frac{1}{R}(kojy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta+jy\bar{a}\ \alpha.jy\bar{a}\ \beta),$$

which is

$$\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi,$$

whence, proceeding as before, we get

$$jy\bar{a}\ (\alpha-\beta)=\frac{1}{R}\ (jy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta-kojy\bar{a}\ \alpha.jy\bar{a}\ \beta)$$

or $\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$.

Alternative Proof. Let the arc YP=arc $PP_1=\beta$ and the arc YQ=arc $QQ_1=\alpha$. Then it is obvious from the figure that

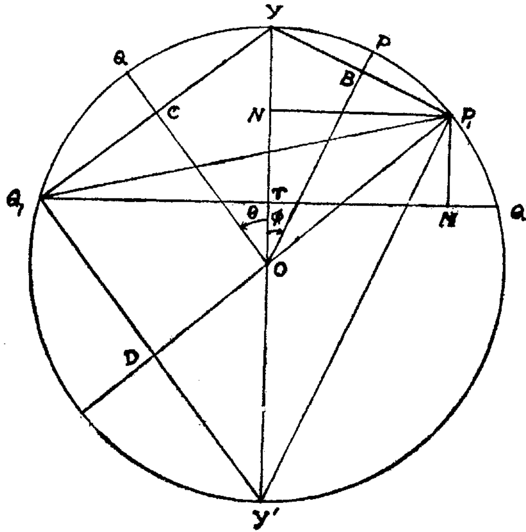


Fig. 8

$YP_1=2\ jy\bar{a}\ \beta$, $Y'P_1=2\ kojy\bar{a}\ \beta$,

$YQ_1=2\ jy\bar{a}\ \alpha$, $Y'Q_1=2\ kojy\bar{a}.\alpha$.

Also

$$Q_1P_1^2=Q_1D^2+DP_1^2=\{jy\bar{a}\ (2\alpha+2\beta)\}^2+\{(utjy\bar{a}\ (2\alpha+2\beta)\}^2.$$

Therefore, $Q_1P_1=2\ jy\bar{a}\ (\alpha+\beta)$.

Similarly $Q'_1P_1=2\ jy\bar{a}\ (\alpha-\beta)$.

By the geometrical rules for finding the height and the segments of the base of a triangle whose sides are known, it can be easily proved that

$$Y'N=\frac{2}{R}\ (kojy\bar{a}\ \beta)^2,\ YN=\frac{2}{R}\ (jy\bar{a}\ \beta)^2,\ P_1N=\frac{2}{R}\ jy\bar{a}\ \beta.\ kojy\bar{a}\ \beta,$$

$$Y'T=\frac{2}{R}\ (kojy\bar{a}\ \alpha)^2,\ \ YT=\frac{2}{R}(jy\bar{a}\ \alpha)^2,\ QT=\frac{2}{R}\ 2\ jy\bar{a}\ \alpha.\ kojy\bar{a}\ \alpha.$$

Now $Q_1P_1^2 = Q_1M^2+P_1M^2$
$= (Q_1T+P_1N)^2+(YT-YN)^2$

$$= \frac{4}{R^2}(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \alpha+jy\bar{a}\ \beta.\ kojy\bar{a}\ \beta)^2+\frac{4}{R^2}\{(jy\bar{a}\ \alpha)^2-(jy\bar{a}\ \beta)^2\}^2$$

$$= \frac{4}{R^2}(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta+kojy\bar{a}\ \alpha.\ jy\bar{a}\ \beta)^2$$

$$\therefore \quad jy\bar{a}\ (\alpha+\beta)= \frac{1}{R}(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta+kojy\bar{a}\ \alpha.jy\bar{a}\ \beta),$$

which is equivalent to
$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$.

Again

$Q'P_1^2 = Q'M^2+MP_1^2$
$= (Q_1T-P_1N)^2+(YT-YN)^2$

$$=\frac{4}{R^2}(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \alpha-jy\bar{a}\ \beta.\ kojy\bar{a}\ \beta)^2+\frac{4}{R^2}\{(jy\bar{a}\ \alpha)^2-(jy\bar{a}\ \beta)^2\}^2$$

whence

$$jy\bar{a}\ (\alpha-\beta)=\frac{1}{R}(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \beta-kojy\bar{a}\ \alpha.\ jy\bar{a}\ \beta),$$

which is equivalent to
$\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$.

The above theorems are called *Bhāvanā* ("demonstration" or "proof" meaning "any thing demonstrated or proved", hence "theorem").

They are again divided into *Samāsa-bhāvanā* or *Yoga-bhāvanā* ("Addition Theorem") and *Antara-bhāvanā* or *Viyoga-bhāvanā* ("Subtraction Theorem").[1]

In the proofs given above the arcs α and β have been tacitly assumed to be each less than 90°. But the theorems are quite general and hold true even when the arcs are greater than 90°.

Thus Kamalākara observes:

"Even when the two arcs go beyond 90° to any even or odd quadrant (the theorems) will remain the same, not otherwise. That is the opinion of those who are aware of the true facts."[2]

[1] *SiŚi, Gola,* xiv. 21 (*Gloss*); *SiTVi,* ii. 65.
[2] *SiTVi,* ii. 66f.

Functions of Multiple Angles

As corollaries to the general case of the theorems for expanding sin $(\theta \pm \phi)$ and cos $(\theta \pm \phi)$, Bhāskara II (1150) indicates how to derive the functions of multiple angles. He observes:

"This being proved, it becomes an argument for determining the values of other functions. For example, take the case of the combination of functions of equal arcs: by combining the functions of any arc with those of itself, we get the functions of twice that arc; by combining the functions of twice the arc with those of twice the arc, we get functions of four times that arc; and so on. Next take the case of combination of functions of unequal arcs: on combining the functions of twice an arc with those of thrice that arc, by the addition theorem we get the functions of five times that arc; but by the subtraction theorem, we get the functions of one time that arc; and so on."[1]

The theorems meant here are clearly these:

$$\sin 2\theta = 2 \sin \theta \cos \theta,$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta,$$
$$\sin 4\theta = 2 \sin 2\theta \cos 2\theta,$$
$$= 4 \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta),$$
$$\cos 4\theta = \cos^2 2\theta - \sin^2 2\theta,$$
$$= \cos^4 \theta - 6 \sin^2\theta\cos^2\theta + \sin^4\theta,$$
$$\sin 3\theta = 3 \sin\theta \cos^2\theta - \sin^3\theta,$$
$$\cos 3\theta = \cos^3\theta - 3 \cos\theta \sin^2\theta,$$
$$\sin 5\theta = \sin 2\theta \cos 3\theta + \cos 2\theta \sin 3\theta,$$
$$\cos 5\theta = \cos 2\theta \cos 3\theta - \sin 2\theta \sin 3\theta,$$
$$\sin\theta = \sin 3\theta \cos 2\theta - \cos 3\theta \sin 2\theta,$$
$$\cos\theta = \cos 3\theta \cos 2\theta + \sin 3\theta \sin 2\theta.$$

All these theorems have been expressly stated by Kamalākara (1658).

He says:

"Hereafter I shall describe how to find the *R*sine of twice, thrice, four times or five times an arc, having known the *R*sine of the sum of two arcs. The product of the *R*sine and *R*cosine of an arc is multiplied by 2 and divided by the radius; the result is the *R*sine of twice that arc."[2]

"The difference of the squares of the *R*sine and *R*cosine of an arc is divided by the radius; the quotient is certainly the *R*cosine of twice that arc."[3]

He has given the following proof of the above two formulae.[4]

---

[1] *SiŚi, Gola*, xiv. 21-2 (*Gloss*).
[2] *SiTVi*, ii. 73.
[3] *Ibid*, ii. 90.
[4] See his own gloss on the preceding rules.

Let the arc XP=arc PQ, then $X'Q=2OC=2$ *kojyā* α. Now from the right angled triangles OPM, $X'QT$, we have

$PO : PM :: X'Q : QT$,
and $OP : OM :: X'Q : X'T$.
$\therefore$ $OP.QT=PM.X'Q=2PM.OC$,
and $OP.X'T=OM.X'Q$,
or $OP(X'O+OT)=OM.\ 2OC=2OM^2$, because $OC=OM$
or $OP(OP+OT)=2OM^2$,
or $OP.OT=2OM^2-OP^2=OM^2-PM^2$.

Therefore,

$$QT=\frac{2\,PM.OC}{R}$$

$$OT=\frac{OM^2-PM^2}{R}.$$

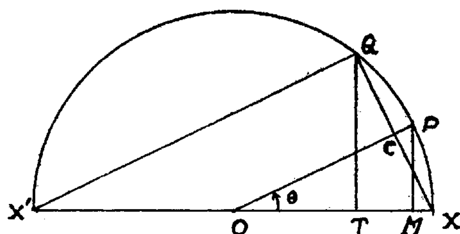


Fig. 9

Hence,

$$jy\bar{a}\ 2\alpha=\frac{2jy\bar{a}\ \alpha .\ kojy\bar{a}\ \alpha}{R}.$$

$$kojy\bar{a}\ 2\alpha=\frac{(kojy\bar{a}\ \alpha)^2-(jy\bar{a}\ \alpha)^2}{R}.$$

That is,

$$\sin 2\theta = 2\sin\theta\cos\theta,$$
$$\cos 2\theta = \cos^2\theta-\sin^2\theta.$$

It has been further observed that these results can be easily deduced from the Addition Theorem by putting $\phi=\theta$.

"The sine of an arc is divided by the sine of one sign (i.e. 30°); the square of the quotient is subtracted from 3 and the remainder is multiplied by the R sine of the arc; the result is the R sine of thrice that arc."[1]

$$jy\bar{a}\ 3\alpha=jy\bar{a}\ \alpha\left\{3-\left(\frac{jya\ \alpha}{jy\bar{a}\ 30^\circ}\right)^2\right\}.$$

That is, $\sin 3\theta=\sin\theta\left(3-\frac{\sin^2\theta}{\sin^2 30^\circ}\right)$.

[1] *SiTVi*, ii. 74; also the Gloss.

By the successive application of the Addition Theorems, Kamalākara obtains the formulae:[1]

$$\begin{aligned}
jy\bar{a}\ 3\alpha &= \{3R^2\, jy\bar{a}\ \alpha - 4(jy\bar{a}\ \alpha)^3\}/R^2,\\
kojy\bar{a}\ 3\alpha &= \{4\ (kojy\bar{a}\ \alpha)^3 - 3R^2\ kojy\bar{a}\ \alpha\}/R^2,\\
jy\bar{a}\ 4\alpha &= 4\ \{(kojy\bar{a}\ \alpha)^3\ jy\bar{a}\ \alpha - (jy\bar{a}\ \alpha)^3\ kojy\bar{a}\ \alpha\}/R^3,\\
kojy\bar{a}\ 4\alpha &= \{(kojy\bar{a}\ \alpha)^4 - 6\ (kojy\bar{a}\alpha)^2\ (jy\bar{a}\alpha)^2 + (jy\bar{a}\ \alpha)^4\}/R^3,\\
jy\bar{a}\ 5\alpha &= \{(jy\bar{a}\ \alpha)^5 - 10\ (jy\bar{a}\ \alpha)^3\ (kojy\bar{a}\ \alpha)^2 + 5\ jy\bar{a}\ \alpha\ (kojy\bar{a}\ \alpha)^4\}/R^4,\\
kojy\bar{a}\ 5\alpha &= \{(kojy\bar{a}\ \alpha)^5 - 10\ (kojy\bar{a}\ \alpha)^3\ (jy\bar{a}\ \alpha)^2 + 5\ kojy\bar{a}\ \alpha\ (jy\bar{a}\ \alpha)^4\}/R^4;
\end{aligned}$$

which are of course equivalent to

$$\begin{aligned}
\sin 3\theta &= 3 \sin \theta - 4 \sin^3 \theta,\\
\cos 3\theta &= 4 \cos^3 \theta - 3 \cos \theta,\\
\sin 4\theta &= 4\ (\cos^3 \theta \sin \theta - \sin^3 \theta \cos \theta),\\
\cos 4\theta &= \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta,\\
\sin 5\theta &= \sin^5 \theta - 10 \sin^3\theta \cos^2 \theta + 5 \sin \theta \cos^4 \theta,\\
\cos 5\theta &= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta.
\end{aligned}$$

Functions of Submultiple Angles

It has been stated before that the following two formulae for the sine of half an angle were known to almost all the Hindu astronomers:

$$\sin \frac{\theta}{2} = \tfrac{1}{2}\sqrt{\sin^2\theta + \operatorname{versin}^2\theta},$$

$$\sin \frac{\theta}{2} = \sqrt{\tfrac{1}{2}\ (1 - \cos\theta)}.$$

Besides these[2] Kamalākara has given formulae for the functions of the third, fourth and fifth parts of an arc.

"Find the cube of one-third the *R*sine of an arc; divide it by the square of the radius; the quotient is added to its one-third and the sum again to one-third the *R*sine of the arc; the result is nearly the *R*sine of one-third that arc. From the cube of this again further accurate values can be obtained."[3]

$$jy\bar{a}\ \frac{\alpha}{3} = \tfrac{1}{3}\, jy\bar{a}\ \alpha + \frac{4}{3\ R^2}\left(\frac{jy\bar{a}\ \alpha}{3}\right)^3.$$

The *rationale* of this formula has been stated to be this: As has been proved before

$$jy\bar{a}\ 3\beta = 3\ jy\bar{a}\,\beta - \frac{4}{R^2}\ (\,jy\bar{a}\ \beta\,)^3.$$

[1] *SiTVi*, ii. 75-7 and also the Gloss on them.
[2] *SiTVi*, ii. 78 f.
[3] *Ibid*, ii. 81.

Put $3\beta=\alpha$; then this formula will become

$$jy\bar{a}\,\frac{\alpha}{3}=\tfrac{1}{3}\,jy\bar{a}\alpha+\frac{4}{3R^2}\left(jy\bar{a}\frac{\alpha}{3}\right)^3. \qquad (i)$$

Now $jy\bar{a}\,\frac{\alpha}{3}$ can be taken, says Kamalàkara, as a *rough approximation* (*sthūla*) to be equal to $\left(\frac{jy\bar{a}\,\alpha}{3}\right)^3$. So that approximately

$$jy\bar{a}\,\frac{\alpha}{3}=\tfrac{1}{3}\,jy\bar{a}\,\alpha+\frac{4}{3R^2}\left(\frac{jy\bar{a}\,\alpha}{3}\right)^3, \qquad (ii)$$

as stated in the rule. Very nearer approximation (*sūkṣmāsanna*) to the value of $jy\bar{a}\,\frac{\alpha}{3}$ can be found by substituting the cube of this value in the last term of (*i*) and by repeating similar operations.

The form (*ii*) is equivalent to

$$\sin\frac{\theta}{3}=\frac{1}{3}\sin\theta+\frac{4}{81}\sin^3\theta.$$

"From the known value of the *R*sine of an arc, first calculate the value of the *R*sine of half that arc; the *R*sine of the arc is divided by that and multiplied by the square of the radius; the result is subtracted from twice the square of the radius. Half the square-root of the remainder is the value of the *R*sine of one-fourth that arc."[1]

$$jy\bar{a}\,\frac{\alpha}{4}=\frac{1}{2}\sqrt{2R^2-R^2\,\frac{jy\bar{a}\,\alpha}{jy\bar{a}\,(\alpha/2)}}\,.$$

The *rationale* of this formula is given thus: It is known that

$$jy\bar{a}\,4\beta=\frac{4}{R^3}\left\{(kojy\bar{a}\,\beta)^3\,jy\bar{a}\,\beta-(jy\bar{a}\,\beta)^3\,kojy\bar{a}\,\beta\right\},$$

$$=\frac{4}{R^3}\left\{R^2\,jy\bar{a}\,\beta\,kojy\bar{a}\,\beta-2\,(jy\bar{a}\beta)^3\,kojy\bar{a}\,\beta\right\},$$

Putting α for 4β, we get

$$R^3\,jy\bar{a}\,\alpha=4\,jy\bar{a}\,\frac{\alpha}{4}\,kojy\bar{a}\,\frac{\alpha}{4}\left\{R^2-2\left(jy\bar{a}\,\frac{\alpha}{4}\right)^2\right\},$$

$$=2R\,jy\bar{a}\,\frac{\phi}{2}\left\{R^2-2\left(jy\bar{a}\,\frac{\alpha}{4}\right)^2\right\};$$

[1] *SiTVi*, ii. 82-83.

whence

$$jy\bar{a}\,\frac{\alpha}{4}=\frac{1}{2}\sqrt{2R^2-R^2\,(jy\bar{a}\,\alpha)/jy\bar{a}\,\frac{\alpha}{2}}$$

or

$$\sin\frac{\theta}{4}=\frac{1}{2}\sqrt{2-\frac{\sin\theta}{\sin(\theta/2)}}.$$

"The intelligent should first find the one-fifth of the *R*sine of the given arc; divide four times the cube of that by the square of the radius; the quotient should be called the "first". Multiply the "first" by the square of the fifth part of the *R*sine and divide the product by the square of the radius; lessen this quotient by its fifth part and mark the remainder as the "second". One-fifth of the *R*sine of the arc added with the "first" and diminished by the "second", will be clearly the value of the *R*sine of the fifth part of the arc. Finding the value of the "first" again from this, further approximate value to the *R* sine of one-fifth the arc can be found. Still closer approximations can be obtained by repeating the process stated above."[1]

$$jy\bar{a}\,\frac{\alpha}{5}=\frac{1}{5}\,jy\bar{a}\,\alpha+\frac{4}{R^2}\left(\frac{jya\,\alpha}{5}\right)^3-\frac{16}{5R^4}\left(\frac{jy\breve{a}\,\alpha}{5}\right)^5.$$

The *rationale* is stated to be this: It has been established before that
$R^4\;jy\bar{a}\;5\beta=(jy\bar{a}\;\beta)^5-10\,(jy\bar{a}\;\beta)^3\;(kojy\bar{a}\;\beta)^2+5jy\bar{a}\;\beta\;(kojy\bar{a}\;\beta)^4.$

Substituting the value $R^2-(jy\bar{a}\;\beta)^2$ for $(kojy\bar{a}\;\beta)^2$ in this, we get
$R^4\;jy\bar{a}\,5\,\beta=16\,(jy\bar{a}\;\beta)^5-20R^2\;(jy\bar{a}\;\beta)^3+5R^4\;jy\bar{a}\;\beta$

Putting α for 5β,

$$jy\bar{a}\,\frac{\alpha}{5}=\frac{1}{5}\,jy\bar{a}\,\alpha+\frac{4}{R^2}(jy\bar{a}\,\frac{\alpha^3}{5})-\frac{61}{5R^4}\,(jy\bar{a}\,\frac{\alpha}{5})^5. \qquad (i)$$

In the last two terms on the right hand side, one may take as a rough approximation

$$jy\bar{a}\,\frac{\alpha}{5}=\frac{1}{5}\,jy\bar{a}\,\alpha;$$

so that

$$jy\bar{a}\,\frac{\alpha}{5}=\frac{1}{5}\,jy\bar{a}\alpha+\frac{4}{R^2}\left(\frac{jy\bar{a}\,\alpha}{5}\right)^3-\frac{16}{5R^4}\left(\frac{jy\bar{a}\,\alpha}{5}\right)^5\;. \qquad (ii)$$

[1] *SiTVi*, ii. 84-87.

Again substituting this value of *jyā* $\frac{\alpha}{5}$ in the last two terms of (*i*) and repeating similar operations, closer approximations to the value of *jyā* $\frac{\alpha}{5}$ can be obtained.

The formula (*ii*) is equivalent to

$$\sin\frac{\theta}{5}=\frac{1}{5}\sin\theta+4\left(\frac{\sin\theta}{5}\right)^3-\frac{16}{5}\left(\frac{\sin\theta}{5}\right)^5.$$

Kamalākara then observes that "in this way, the *R*sines of other desired submultiples of an arc should be obtained."[1]

$Sin\ \frac{(\theta-\phi)}{2}$.

Bhāskara II says:

"Find the difference of the *R*sines of two arcs and also of their *R*cosines; then find the square-root of the sum of the squares of the two results; half this root will be the *R*sine of half the difference of the two arcs."[2]

That is,

$$jy\bar{a}\ \frac{1}{2}(\alpha-\beta)=\frac{1}{2}\{(jy\bar{a}\ \alpha-jy\bar{a}\ \beta)^2+(kojy\bar{a}\ \alpha-kojy\bar{a}\ \beta)^2\}^{\frac{1}{2}}.$$

or, in modern notations,

$$\sin\tfrac{1}{2}(\theta-\phi)=\tfrac{1}{2}\{(\sin\theta-\sin\phi)^2+(\cos\theta-\cos\phi)^2\}^{\frac{1}{2}}.$$

Kamalākara writes:

"Half the square-root of the sum of the squares of the differences of *R*sines and *R*cosines of two arcs is certainly equal to the *R*sine of half the difference of the two arcs."[3]

The latter has given the following proof of it.[4] Let the arc XP be denoted by β and the arc XQ by α; then

[1] *SiTVi*, ii. 87 (c-d).
[2] *SiŚi, Gola*, xiv. 13.
[3] *SiTVi*, ii. 94.
[4] *Ibid*, (Gloss).

$$QC = QT - PM = jy\bar{a}\ \alpha - jy\bar{a}\ \beta,$$
$$PC = OM - OT = kojy\bar{a}\ \beta - kojy\bar{a}\ \alpha.$$

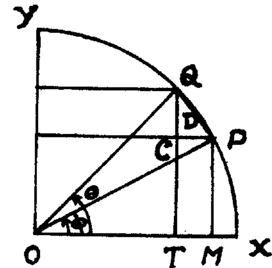


Fig.10

Now,

$$PQ^2 = QC^2 + PC^2.$$

Hence,

$$jy\bar{a}\ \tfrac{1}{2}\ (\alpha - \beta) = \tfrac{1}{2}\left\{(jy\bar{a}\ \alpha - jy\bar{a}\ \beta)^2 + (kojy\bar{a}\ \alpha - kojy\bar{a}\ \beta))^2\right\}^{\frac{1}{2}};$$

which is equivalent to

$$\sin \tfrac{1}{2}\ (\theta - \phi) = \tfrac{1}{2}\left\{(\sin\ \theta - \sin\phi)^2 + (\cos\ \theta - \cos\phi)^2\right\}^{\frac{1}{2}}.$$

Theorem of Sines

Brahmagupta[1] has made use of the important relation

$$\frac{a}{jy\bar{a}\ A} = \frac{b}{jy\bar{a}\ B} = \frac{c}{jy\bar{a}\ C}$$

which is of course equivalent to

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

between the sides (a, b, c) and angles (A, B, C) of a plane triangle.

4. Functions of Particular Angles

Sine of 30°, 45° and 60°. Preliminary to the calculation of tables of trigonometrical functions almost all the Hindu writers have stated the values of the *R*sines of 30°, 45° and 60°.

$jy\bar{a}\ 30° = \sqrt{R^2/4}$, $jy\bar{a}\ 45° = \sqrt{R^2/2}$, $jy\bar{a}\ 60° = \sqrt{3R^2/4}$ or, in modern notations, $\sin 30° = \frac{1}{2}$, $\sin 45° = 1/\sqrt{2}$, $\sin 60° = \frac{1}{2}\sqrt{3}$.

Śrīpati indicates the proof thus:

"The experts in spherics say that the circum-radius of a regular hexagon is equal to a side. So it will be perceived that the chord of the sixth part of the circumference

[1]*KK*, Part I, viii. 2. Our attention to this was first drawn by Professor P. C. Sen Gupta, who was then preparing a new edition of *Khaṇḍakhādyaka* with English translation and critical notes. This rule occurs in the works of other Hindu astronomers also.

of a circle is equal to its semi-diameter. The hypotenuse arising from the base and perpendicular (of a right-angled triangle) each equal to the semi-diameter is the chord of the fourth part of the circumference. Half those (chords) will be the *R*sines of half those arcs."[1]

The same proof is also given by Bhāskara II:[2]

"The side of a regular hexagon inscribed in a circle is equal to its radius; this is well known and has also been stated in (my) Arithmetic. Hence follows that the *R*sine of 30° is half the radius."

"Suppose a right-angled triangle whose base and perpendicular are each equal to the radius; the square-root of the sum of the squares of these will be equal to the side of a square inscribed in that circle and it is again the chord of 90°. Take the half of that. Hence the sum of the squares (of the sides) is divided by four; and the result is half the square of the radius. The square-root of that, it thus follows, is the *R*sine of 45°."

"The *R*sine of 60° is equal to the *R*cosine of 30°, the *R*sine of which is equal to half the semi-diameter."

Sin 18° and sin 36°. Bhāskara II says:

"The square-root of five times the square of the radius is diminished by the radius and the remainder is divided by four; the result is the exact value of the *R*sine of 18°."[3]

$$jy\bar{a}\ 18^\circ = \tfrac{1}{4}\left(\sqrt{5R^2} - R\right);$$

or $$\sin 18^\circ = \tfrac{1}{4}(\sqrt{5} - 1).$$

"The square-root of five times the square of the square of the radius is subtracted from five times the square of the radius and the remainder is divided by eight; the square-root of the quotient is the *R* sine of 36°."

"Or the radius multiplied by 5878 and divided by 10000, is the *R*sine of 36°. The *R*cosine of that is the *R*sine of 54°."[4]

$$jy\bar{a}\ 36^\circ = \sqrt{\tfrac{1}{8}\left(5R^2 - \sqrt{5R^4}\right)} = \frac{5878R}{10000}.$$

That is, $$\sin 36^\circ = \sqrt{\tfrac{1}{8}\left(5 - \sqrt{5}\right)} = \frac{5878}{10000}$$

[1] *SiŚe*, xvi. 11-2.
[2] *SiŚi, Gola*, v. 3-4 (Gloss).
[3] *Ibid, Gola*, xiv. 9.
[4] *Ibid, Gola*, xiv. 7-8.

Since $\sqrt{5} = 2{\cdot}237411$ approximately

$\therefore \quad 5 - \sqrt{5} = 2{\cdot}762589 \ldots$

$\therefore \quad \sqrt{\frac{1}{8}(5-\sqrt{5})} = \sqrt{.345323 \ldots} = {\cdot}5878$ approximately.

Kamalākara proved the results thus:

Let x denote *jyā* 18°; then

$\frac{1}{2} R (R-x) = \frac{1}{2} R.\ utjy\bar{a}\ 72^\circ = (jy\bar{a}\ 36^\circ)^2;$

$\frac{2x^2}{R} = R - kojy\bar{a}\ 36^\circ = utjy\bar{a}\ 36^\circ;$

$$\frac{1}{2} R (R-x) + \left(\frac{2x^2}{R}\right)^2 = (jy\bar{a}\ 36^\circ)^2 + (utjy\bar{a}\ 36^\circ)^2$$
$$= 4\,(jy\bar{a}\ 18^\circ)^2$$
$$= 4x^2,$$

or $8\,x^2\,R^2 = 8\,x^4 - R^3\,x + R^4$,

or, multiplying by 8 and arranging,

$16R^2\,x^2 + 8\,R^3 x + R^4 = 9\,R^4 - 48\,R^2\,x^2 + 64\,x^4,$

whence taking the positive square roots of the two sides, we get

$4\,Rx + R^2 = 3\,R^2 - 8\,x^2,$
or $(4\,x + R)^2 = 5R^2$.

Therefore $x = \frac{1}{4}(\sqrt{5R^2} - R)$;

the other sign is neglected since x must be less than R.

Again $(jy\bar{a}\ 36^\circ)^2 = \frac{1}{2}R(R-x)$,

$$= \frac{R}{8}(5R - \sqrt{5\,R^2}),$$
$$\therefore jy\bar{a}\ 36^\circ = \sqrt{\tfrac{1}{8}(5R^2 - \sqrt{5R^4})}$$

Sin π/N

In his treatise on arithmetic, Bhāskara II has given a rule which yields the *R*sine of certain particular angles to a very fair degree of approximation.

"Multiply the diameter of a circle by 103923, 84853, 70534, 60000, 52055, 45922 and 41031 severally and divide the products by 120000; the quotients will be the sides of regular polygons inscribed in the circle from the triangle to the enneagon respectively."[1]

If S_n be a side of a regular polygon of n sides inscribed in a circle of diameter D, then according to Bhāskara II,

$$S_3 = D\,\frac{103923}{120000} = D \times \cdot 866025$$

$$S_4 = D\,\frac{84853}{120000} = D \times \cdot 707108\dot{3}$$

$$S_5 = D\,\frac{70534}{120000} = D \times \cdot 58778\dot{3}$$

$$S_6 = D\,\frac{60000}{120000} = D \times \cdot 5$$

$$S_7 = D\,\frac{52055}{120000} = D \times \cdot 433791\dot{6}$$

$$S_8 = D\,\frac{45922}{120000} = D \times \cdot 38268\dot{3}$$

$$S_9 = D\,\frac{41031}{120000} = D \times \cdot 341925$$

where are given the formulae of Bhāskara II first in their original forms and then in decimals. Now, we know that

$$s_n = D \sin \frac{\pi}{n}\,.$$

Hence it is found that

$$\left.\begin{array}{l} \sin 60^\circ = \cdot 866025 \\ \sin 45^\circ = \cdot 707108\dot{3} \\ \sin 36^\circ = \cdot 58778\dot{3} \end{array}\right\} \quad \begin{array}{l} \sin \pi/7 = \cdot 433791\dot{6} \\ \sin \pi/8 = \cdot 38268\dot{3} \\ \sin \pi/9 = \cdot 341925 \end{array}$$

According to modern computation

$$\left.\begin{array}{l} \sin 60^\circ = \cdot 8660254\ldots. \\ \sin 45^\circ = \cdot 7071067\ldots. \\ \sin 36^\circ = \cdot 5877853 \end{array}\right\} \quad \begin{array}{l} \sin \pi/7 = \cdot 4338819 \\ \sin \pi/8 = \cdot 3826834 \\ \sin \pi/9 = \cdot 3420201 \end{array}$$

[1] *L*, vss. 206-7, p. 207.

Comparing the two tables we find that except in case of sin $\pi/7$ and sin $\pi/9$ Bhāskara'a approximations are correct up to five places of decimals; in these two latter cases the results are near enough.

Approximate Formula of Bhāskara I

Bhāskara I (629) has given the following rule for the calculation of the *R*sine and *R*cosine of an arc without the help of a table.

"Subtract the arc in degrees from the degrees of the semi-circumference and multiplying the arc by the remainder, put down (the result) at two places, (at one place) subtract (the quantity) from 40500; by one-fourth of the ıemainder divide the quantity (at the second place) multiplied by the maximum value of the function; thus the value of the direct or reversed *R*sine of an arc and its complement is obtained wholly."[1]

If α be an arc of a circle of radius R in terms of degrees, then

$$jy\bar{a}\,\alpha = \frac{R(C/2-\alpha)\alpha}{\{40500-(C/2-\alpha)\,\alpha\}/4},$$

where C denotes the circumference of the circle in terms of degrees. Since $40500=(5/4) \times 180 \times 180$, we can write the formula in the form

$$jy\bar{a}\,\alpha = \frac{4R(C/2-\alpha)\alpha}{5/4\,(C/2)^2-(C/2-\alpha)\,\alpha},$$

which is of course equivalent to

$$\sin\,\theta = \frac{4\,(\pi-\theta)\,\theta}{(5/4)\,\pi^2-(\pi-\theta)\,\theta}.$$

From a statement of Bhāskara I it appears that this formula was known to Ārya-bhaṭa I.[2]

The above formula has been restated by Brahmagupta (628) thus:

"Subtract the degrees of an arc or its complement from the semicircle (i.e. 180) and multiply (the remainder) by that; subtract one-fourth the product from 10125; divide the product by the remainder and multiply by the semi-diameter; (the result) is the *R*sine of that (arc or its complement)."[3]

$$jy\bar{a}\,\alpha = \frac{R\,(180-\alpha)\,\alpha}{10125-(180-\alpha)\,\alpha/4}.$$

[1] *MBh*, vii. 17ff.
[2] Bhāskara I's com. on *Ā*, i. 11, p. 40.
[3] *BrSpSi*, xiv. 23.

Almost in the same way Śrīpati (1039) says:

"Subtract the degrees of an arc or its complement from 180 and multiply (the remainder) by that; subtract one-fourth the product from 10125; multiply the product by the semi-diameter and divide by this remainder; thus the *R*sine of an arc or its complement can be found even without (a table of *R*sines)."[1]

Bhāskara II (1150) writes:

"Subtract an arc from the circumference and multiply (the remainder) by the arc; this product is called the 'first'. From five times the fourth part of the square of the circumference subtract the 'first', and by the remainder divide the 'first' multiplied by four times the diameter; the quotient will be the chord of the arc."[2]

If s denote the chord of an arc β of a circle, then

$$s = \frac{8R\,(C-\beta)\beta}{\frac{5}{4}C^2-(C-\beta)\beta}.$$

Now if $\beta=2\alpha$, then $s=2\,jy\bar{a}\,\alpha$. So that on making the substitutions this formula will easily reduce to that of the elder Bhāskara.

This formula has been used by Gaṇeśa (1545) in his *Grahalāghava*.[3] Though it gives only a roughly approximate (*sthūla*) value of the *R*sine of an arc, observes Bhāskara II, it simplifies operations.

On putting $\theta = \pi/2-\phi$, in the above approximate formula, it becomes

$$\cos\phi = \frac{16\,(\pi/2+\phi)\,(\pi/2-\phi)}{5\pi^2-4\,(\pi/2+\phi)\,(\pi/2-\phi)}$$

$$= \frac{\pi^2-4\,\phi^2}{\pi^2+\phi^2}$$

$$= \left(1-\frac{4\phi^2}{\pi^2}\right)\left(1-\frac{\phi^2}{\pi^2}+\frac{\phi^4}{\pi^4}\right),$$

neglecting higher powers. Therefore, to the same order of approximation,

$$\cos\phi = 1-\frac{5\phi^2}{\pi^2}+\frac{5\phi^4}{\pi^4}.$$

[1] *SiŚe*, iii. 17.
[2] *L*, vs. 210, p. 21e. Also see *GK*, par 2, pp. 80-81.
[3] *GrL*, ii. 2 f.

If we put $\pi = \sqrt{10}$ approximately, we get

$$\cos\phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{20}.$$

nearly. According to modern Trigonometry, to the same order of approximation,

$$\cos\phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{24}.$$

Again putting $\phi = \pi/n$ in Bhāskara I's formula, where n is an integer, we get

$$\sin \pi/n = \frac{16\,(n-1)}{5n^2 - 4(n-1)}$$

whence we have

$\sin \pi/7 = \cdot 4343 \ldots,\quad \sin \pi/8 = \cdot 3835 \ldots,\quad \sin \pi/9 = \cdot 3431 \ldots,$
which are correct up to two places of decimals, the third figure in every case being too large.

Inverse Formula of Brahmagupta

Brahmagupta gave the following rule for finding approximately the arc corresponding to a given *R*sine function:

"Multiply 10125 by the given *R*sine and divide by the quarter of the given *R*sine plus the radius; subtracting the quotient from the square of 90, extract the square-root and subtract (the root) from 90; the remainder will be in degrees and minutes; thus will be found the arc of the given *R*sine without the table of *R*sines."[1]

If α be the arc corresponding to the given *R*sine function m, then the rule says that

$$\alpha = 90 - \sqrt{8100 - \frac{10125m}{(m/4+r)}}\ .$$

This result follows easily on reversing the approximate formula for the *R*sine and was very likely obtained in the same way.

$$m = jy\bar{a}\,\alpha = \frac{R(180-\alpha)\,\alpha}{10125-(180-\alpha)\,\alpha/4}\ .$$

Then,

$$\alpha^2 - 180\alpha + \frac{10125m}{(m/4+r)} = 0.$$

[1]*BrSpSi*, xiv. 25-6.

Therefore,

$$\alpha = 90 - \sqrt{8100 - \frac{10125m}{(m/4+r)}}\,.$$

The negative sign of the radical being retained, since α is supposed to be less than 90°.

Śrīpati describes the inverse formula thus:

"Multiply 10125 by the given *R*sine and divide by the quarter of the given *R*sine plus the radius; then subtract the quotient from the square of 90; ninety degrees lessened by the square root (of the remainder) will be the arc (determined) without the table of *R*sines."[1]

Bhāskara II writes:

"By four times the diameter added with the chord divide the square of the circumference multiplied by five times a quarter of the chord; the quotient being subtracted from the fourth part of the square of the circumference, and the square-root of the remainder being diminished from half the circumference, the result will be the arc."[2]

That is:

$$\beta = \frac{C}{2} - \left\{ \frac{C^2}{4} - \frac{5s\,C^2}{4(8R+s)} \right\}^{\frac{1}{2}}$$

which follows at once from his form of the approximate formula for the chord s.

5. Trigonometrical Tables

Twenty-four Sines

The Hindus generally calculate tables of trigonometrical functions for every arc of 3°45′, or what they call twenty-four *R*sines in a quadrant. In the choice of 24, they seem to have been led by an ancient observation that "the ninety-sixth part of a circle looks (straight) like a rod". Thus Balabhadra (*c.* 700 A.D.) observes, "If anybody asks the reason of this, he must know that each of these *Kardajat* is 1/96 of the circle = 225 minutes (= $3\frac{3}{4}$ degrees). And if we reckon its *R*sine, we find it also to be 225 minutes."[3]

The origin of this idea again lies in the impression that the human eye-sight reaches to a distance of 1/96th part of the circumference of the earth which appears

[1]*SiŚe*, iii. 18.
[2]*L*, vs. 212, p. 216.
[3]Quoted by Al-Bīrūnī in his *India* (Sachau, *Alberuni's India*, I, p. 275). Balabhadra's works are now lost. According to Chambers' *Mathematical Tables*, we find
sin (3° 45′) = ·0654031, tan (3° 45′) = ·0655435, radian (3° 45′) = ·0654498, so that the assumption is fairly accurate.

flat.[1] A more plausible hypothesis about the choice of 3°45′ as the unit will be this: Having determined previously a very accurate value of π it was simply natural for the Hindus to choose the radius of the circle of reference to be 3438′. They also knew that *R*sin 30°=semi-radius=1719′. Starting with this they began to calculate the function of the semi arcs 15°, 7° 30′, 3°45′ with the help of the well known formulae and in so doing they soon found that 3° 45′ is the first whose *R*sine contains the same number of minutes as the arc. So they chose this arc.[2]

Sūrya-Siddhānta

The earliest known Hindu work to contain a table of trigonometrical functions is the *Sūrya-Siddhānta* (*c.* 300 A.D.). It has a table of *R*sines and versed *R*sines for every arc of 3°45′ of a circle of radius 3438′. The method of computation has been indicated to be as follows:

"The eighth part of the number of minutes in a sign (i.e. 225′) is the first *R*sine. It is divided by itself and then diminished by the quotient; the remainder added with the first *R*sine gives the second *R*sine.

"(Any) *R*sine is divided by the first *R*sine and then diminished by the quotient. The remainder added to the difference of that *R*sine and the preceding *R*sine will give the next *R*sine. Thus can be obtained the 24 *R*sines, which are as follows."[3]

Let the arc PQ=arc $QR=\alpha$

Then,

$$RK = OL = OS + SL = QT + SL,$$

and $$NS - SL = 2\,DS = 2\,\frac{OS}{OQ}\,QC$$

Now $CP^2 = QC\,(2\,OQ - QC)$;

$$\therefore \qquad QC = \frac{QP^2}{2\,OQ}.$$

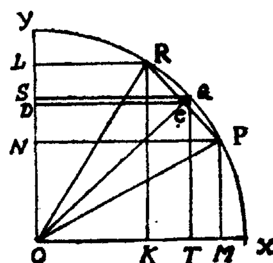


Fig. 11

Hence,

$$NS - SL = OS\left(\frac{QP}{OQ}\right)^2$$

[1]Balabhadra says, "Human eyesight reaches to a point distant from the earth and its rotundity the 96th part of 5000 *yojana*, i.e. 52 *yojana* (exactly $52\frac{1}{12}$). Therefore a man does not observe its rotundity, and hence the discrepancy of opinions on the subject". This remark of Balabhadra has been quoted by Al-Bīrūnī (*India*, I, p. 273).

[2]Cf. *Nouv. Ann. Math.*, xiii (1854), p. 390.

[3]*SūSi*, ii. 15-6.

Or,

$$SL = NS - OS \left(\frac{QP}{OQ}\right)^2$$
$$= (QT - PM) - OS \left(\frac{QP}{OQ}\right)^2$$

Therefore,

$$RK = QT + (QT - PM) - QT \left(\frac{QP}{OQ}\right)^2$$

Now suppose the arc $XQ = n\alpha$; then arc $XP = (n-1)\alpha$; $XR = (n+1)\alpha$;

further $QP = 2jy\bar{a}\frac{\alpha}{2}$. Hence

$$jy\bar{a}\,(n+1)\alpha = jy\bar{a}\;n\alpha + \{\,jy\bar{a}\;n\alpha - jy\bar{a}(n-1)\alpha\,\} - jy\bar{a}\;n\alpha \times \left(\frac{2jy\bar{a}\,\alpha/2}{R}\right)^2,$$

which is equivalent to

$$\sin(n+1)\,\theta = \sin n\theta + \{\sin n\theta - \sin(n-1)\theta\} - \sin n\theta\left(2\sin\frac{\theta}{2}\right)^2.$$

It is also probable that the formula was obtained trigonometrically thus:

$$jy\bar{a}(\,\xi \pm \eta\,) = \frac{1}{R}(jy\bar{a}\;\xi\;kojy\bar{a}\;\eta \pm kojy\bar{a}\;\xi\;jy\bar{a}\;\eta\,).$$

Then,

$$jy\bar{a}\;(\,\xi + \eta\,) - jy\bar{a}\;\xi = \frac{1}{R}(kojy\bar{a}\;\xi\;jy\bar{a}\;\eta - jy\bar{a}\;\xi\;utjy\bar{a}\;\eta\,),$$

and,

$$jy\bar{a}\;\xi - jy\bar{a}\;(\,\xi - \eta\,) = \frac{1}{R}(kojy\bar{a}\;\xi\;jy\bar{a}\;\eta + jy\bar{a}\;\xi\;utjy\bar{a}\;\eta\,).$$

Hence;

$$jy\bar{a}\;(\xi+\eta) - jy\bar{a}\;\xi = jy\bar{a}\;\xi - jy\bar{a}\;(\xi-\eta) - \frac{2\,jy\bar{a}\;\zeta\;utjy\bar{a}\;\eta}{R}$$
$$= jy\bar{a}\;\xi - jy\bar{a}\;(\xi-\eta) - jy\bar{a}\;\xi\left(\frac{2\,jy\bar{a}\frac{\eta}{2}}{R}\right)^2$$

Now put $\eta = \alpha$, $\xi = n\,\alpha$; so that the formula becomes

$$jy\bar{a}\;(n+1)\,\alpha - jy\bar{a}\;n\,\alpha = jy\bar{a}\;n\,\alpha - jy\bar{a}\;(n-1)\,\alpha - jy\bar{a}\;n\,\alpha\left(\frac{2\,jy\bar{a}\frac{\alpha}{2}}{R}\right)^2$$

So far the formula is mathematically accurate. According to the *Sūrya-siddhānta*

$$\measuredangle = 3°45' = 225',\ jy\bar{a}\ \measuredangle = 225',\ R = 3438'$$

Therefore

$$\left(\frac{2\,jy\bar{a}\ \measuredangle/2}{R}\right)^2 = \left(\frac{jy\bar{a}\ \measuredangle}{R}\right)^2 \text{ approximately}$$

$$= \left(\frac{225}{3438}\right)^2 = \left(\frac{1}{15{\cdot}28}\right)^2 = \frac{1}{225} \text{ approximately.}$$

Hence we get

$$\sin (n+1)\ \theta = \sin n\theta + \{\sin n\theta - \sin\ (n-1)\ \theta\} - \frac{\sin n\theta}{225},$$

where

$$\theta = 3°45' \text{ and } n = 1, 2, \ldots, 24.$$

According to modern calculation, the divisor in the last term will be slightly different. For

$$\left(2 \sin \frac{\theta}{2}\right)^2 = (2 \sin 1°52'30'')^2 = {\cdot}00428255 = \frac{1}{233{\cdot}506}, \text{ nearly.}$$

This little discrepancy, however, does not make much difference in the values of the *R*sine functions calculated on the basis of that formula. They are indeed fairly accurate even according to modern calculations except in a few instances.[1]

About this method of constructing the tables of *R* sines, Delambre remarks: "The method is curious; it indicates a method of calculating the table of sines by means of their second differences."[2] He then goes on: "This differential process has not up to now been employed except by Briggs who himself did not know that the constant factor was the square of the chord $\triangle A$ (=3° 45′) or of the interval, and who could not obtain it except by comparing the second differences obtained in a different manner. The Indians also have probably done the same; they obtained the method of differences only from a table calculated previously by a geometric process. Here then is a method which the Indians possessed and which is found neither amongst the Greeks, nor amongst the Arabs."[3]

We do not understand what valid grounds were there for Delambre to suppose that the Hindus discovered the above theorem of *R*sines by inspection after having calculated the table of *R*sines by a different method. For there is absolutely no doubt

[1] *Vide infra.*
[2] Delambre, *Histoire de l'Astronomie Ancienne*, t. 1. Paris, 1817, p. 457.
[3] *Ibid*, p. 459 f.

that the ancient Hindus were in possession of necessary and sufficient equipments to derive it in either of the ways indicated above. It is noteworthy that that theorem has an important geometrical foundation. If there be three arcs of a circle in arithmetical progression the sum of the sines of the two extreme arcs is to the sine of the middle arc as the sine of twice the common difference is to the sine of that difference. For

$$jy\bar{a}\,(\xi+\eta)\; jy\bar{a}\,(\xi-\eta) \equiv 2\,jy\bar{a}\,\xi - \frac{2jy\bar{a}\,\xi\, utjy\bar{a}\,\eta}{R}$$

$$= \frac{2\,jy\bar{a}\,\xi\; kojy\bar{a}\,\eta}{R}.$$

Hence,

$$\frac{jy\bar{a}\,(\xi+\eta) + jy\bar{a}\,(\xi-\eta)}{jy\bar{a}\,\xi} = \frac{2\,kojy\bar{a}\,\eta}{R} = \frac{jy\bar{a}\,2\,\eta}{jy\bar{a}\,\eta}.$$

This very remarkable property of the circle was discovered in Europe by Vieta (1580)[1].

Āryabhaṭa

The trigonometrical table of Āryabhaṭa I (499) contains the differences between the successive *R*sines for arcs of every 3° 45′ of a circle of radius 3438′.[2] His first method of computing it, which is rather cryptic, seems to be the same as that followed by Varāhamihira (*infra*). The other is practically the same as that of the *Sūrya-siddhānta*, though put in a different form. He says:

"Divide a quarter of the circumference of a plane circle (into as many equal parts as desired). From (right) triangles and quadrilaterals (can be obtained) the *R*sines of equal arcs, as many as desired, for (any given) radius."[3]

What is meant by the author is very probably this: If *P* be any point on the arc of the quadrant, draw the perpendiculars *PM* and *PN*; also join *PX*. So that corresponding to *P* we have a rectangle *PMON* and a right-angled triangle *PMX*. Now having given the *R*sine (*PM*) of the arc *XP* ($=\alpha$), we can determine from the rectangle *PMON* the side *PN* which is the sine of the arc ($90° - \alpha$). Having found *PN*, we can calculate *MX*, which is equal to $R - jy\bar{a}\,(90° - \alpha)$. Then in the right-angled triangle *PMX*, we can determine the chord PX. Half of this is $jy\bar{a}\;\alpha/2$. Again

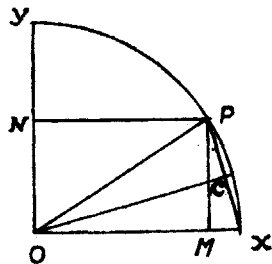


Fig. 12

[1] Playfair, J "Observations on the Trigonometrical Tables of the Brahmins", *Trans. Roy. Soc. Edin*, iv (1798), pp. 83-106; compare also *Asiatic Researches*, iv. p. 165.
[2] *Ā*, i. 12.
[3] *Ibid*, ii. 11.

from a similar set of a rectangle and a right-angled triangle corresponding to the half arc, we can calculate *jyā* $(90°-\alpha/2)$ and *jyā* $\alpha/4$. Proceeding thus we can compute the *R*sines of as many equal arcs as we please and it is clear that in so doing the quadrant will be broken up into a system of right-angled triangles and rectangles, as contemplated in the rule.

This is the interpretation of Āryabhaṭa's rule by his ancient commentators, like Sūryadeva Yajvā and Parameśvara (1430). Another interpretation will be this: The quadrant is trisected by the inscribed equilateral triangle and bisected by the inscribed square. The length of the arc between these points is 15° (=45°—30°). One-fourth of this is 3° 45′. So that the rule under discussion indicates how to divide the quarter of the circumference into portions of 3° 45′ each. If this interpretation is right[1], which is rather forced, then it will have to be said that Āryabhaṭa I gave only one method of computing the trigonometrical table.[2]

The *second* method of Āryabhaṭa I is this:

"The first *R*sine divided by itself and then diminished by the quotient will give the second difference (of tabular *R*sines). For computing any other difference, (the sum of) all the preceding differences is divided by the first *R*sine and the quotient is subtracted from the preceding difference. Thus, all the remaining differences (can be calculated)."[3]

Let $\triangle_1, \triangle_2, \ldots\ldots, \triangle_n$ denote successive differences of the tabular *R*sines, such that, α being equal to 3° 45′,

$$\triangle_1 = jy\bar{a}\ \alpha - jy\bar{a}\ 0,$$
$$\triangle_2 = jy\bar{a}\ 2\alpha - jy\bar{a}\ \alpha,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\triangle_n = jy\bar{a}\ n\alpha - jy\bar{a}\ (n-1)\alpha$$

Then $jya\ n\alpha = \triangle_1 + \triangle_2 + \ldots + \triangle_n$.

The rule says:

$$\triangle_{n+1} = \triangle_n - \frac{\triangle_1 + \triangle_2 + \ldots + \triangle_n}{jy\bar{a}\ \alpha}.$$

On substituting the values, this formula will be found to be equivalent to

$$\{jy\bar{a}\ (n+1)\alpha - jy\bar{a}\ n\alpha\} = \{jy\bar{a}\ n\alpha - jy\bar{a}\ (n-1)\alpha\} - \frac{jya\ n\alpha}{jy\bar{a}\ \alpha}.$$

[1]This interpretation has been suggested by Rodet, Kaye and Sengupta.

[2]In this connection, the reader is referred to "*Āryabhaṭīya* of Āryabhaṭa," edited with English translation by K. S. Shukla and K. V. Sarma, INSA, New Delhi, 1976, pp. 45-51.

[3]*Ā*, ii. 12.

It is also noteworthy that the text also admits of the following interpretation:

"The first *R*sine is divided by itself and then diminished by the quotient; the result with the first *R*sine will give the second *R*sine. For (computing), any of the remaining *R*sines, the sum of all the *R*sines preceding it is divided by the first *R*sine and the quotient is subtracted from the first *R*sine, and the result added to the preceding *R*sine."

$$jy\bar{a}\ (n+1)\alpha = jy\bar{a}\ n\alpha + jy\bar{a}\ \alpha - (jy\bar{a}\ \alpha + jy\bar{a}\ 2\alpha + \ldots\ldots + jy\bar{a}\ n\alpha)/jy\bar{a}\ \alpha.$$

If $\triangle_1, \triangle_2, \ldots.$ be the tabular differences as before, then

$$\triangle_1 - \triangle_2 = \frac{2jy\bar{a}\ \alpha\ (R - kojy\bar{a}\ \alpha)}{R}$$

$$\triangle_2 - \triangle_3 = \frac{2jy\bar{a}\ 2\alpha\ (R - kojy\bar{a}\ \alpha)}{R}$$

....

$$\triangle_n - \triangle_{n+1} = 2\,jy\bar{a}\ n\alpha\ \frac{R - kojy\bar{a}\ \alpha}{R}$$

Adding up, we get

$$\triangle_1 - \triangle_{n+1} = \frac{2(R - kojy\bar{a}\alpha)}{R}\ (jy\bar{a}\ \alpha + jy\bar{a}\ 2\alpha + \ldots\ldots + jy\bar{a}\ n\alpha)$$

Now $\dfrac{2(R - kojy\bar{a})\ \alpha}{R} = \left(\dfrac{2\,jy\bar{a}\ \frac{\alpha}{2}}{R}\right)^2$

$$= \left(\frac{jy\bar{a}\ \alpha}{R}\right)^2 \text{ approximately}$$

$$= \frac{1}{225} \text{ approximately.}$$

Therefore

$$jy\bar{a}\ (n+1)\ \alpha = jy\bar{a}\ n\ \alpha + jy\bar{a}\ \alpha - (jy\bar{a}\ \alpha + jy\bar{a}\ 2\alpha + \ldots\ldots + jy\bar{a}\ n\ \alpha)/225$$

Also

$$\triangle_{n+1} = \triangle_n - \frac{jy\bar{a}\ n\alpha}{225}$$

$$= \triangle_n - \frac{\triangle_1 + \triangle_2 + \ldots\ldots + \triangle_n}{225}.$$

Of these two interpretations the first has been given by the commentator Parameśvara and the second by the commentators Prabhākara, Sūryadeva (b. 1191), Yallaya (1480) and Raghunātharāja (1597).

It should be observed that Āryabhaṭa I does not appear to have used this formula consistently to calculate the whole table. For as will be found from the accompanying table, certain values actually recorded by Āryabhaṭa differ from the values calcu-

TABLE

Differences Δ_n, $n=$	Calculated according to the formula	Recorded by Āryabhaṭa	Calculated according to the modern method
1	225	225	224·856
2	224	224	223·893
3	222·005	222	221·971
4	219·018	219	219·100
5	215·045	215	215·289
6	210·089	210	210·557
7	204·156	205	204·923
8	198·245	199	198·411
9	191·36	191	191·050
10	182·512	183	182·872
11	173·694	174	173·909
12	163·245	164	164·202
13	153·196	154	153·792
14	142·512	143	142·724
15	130·876	131	131·043
16	118·294	119	118·803
17	105·745	106	106·053
18	92·289	93	92·850
19	78·88	79	79·248
20	64·527	65	65·307
21	50·240	51	51·087
22	36·014	37	36·648
23	21·849	22	22·051
24	6·752	7	7·361

lated by the formula. Probably he corrected the calculated values in those cases by comparison with the known values of the sines of 30°, 45°, 60°; or what is much more likely employed the formula only to calculate the *R*sines of intermediate arcs. Other plausible explanations of the discrepancy have been furnished by Krishnaswami Ayyangar[1] and Naraharayya.[2]

Varāhamihira and Lalla

Varāhamihira's (d. 587) table contains the *R*sines for every 3° 45′ and the successive differences of the tabular *R*sines for the radius 60.[3] His method of computation is this:[4] Starting with the known values of *R*sine 30°, *R*sine 45° and *R*sine 60°, by the repeated and proper application of the formulae

$$\sin\frac{\theta}{2} = \tfrac{1}{2}\sqrt{\sin^2\theta + \text{versin}^2\theta}$$

$$\sin\frac{\theta}{2} = \sqrt{\tfrac{1}{2}\text{versin}\,\theta},$$

says he, the other *R*sines may be computed. Lalla[5] gives a table of *R*sines and versed *R*sines for the radius 3438′. His method of computation is the same as that of Āryabhaṭa I and the *Sūrya-siddhānta*. He has also a shorter table of *R*sines and their differences for intervals of 10° of arcs of a circle of radius 150.[6]

Brahmagupta

Brahmagupta (628) takes the radius quite arbitrarily to be 3270. His explanation[7] for this departure from the usual practice is unsatisfactory.[8] He has, however, indicated two methods of computation.[9] One is *graphic* and the other *mathematical.*

Graphic Method. "Starting from the joint of two quadrants, mark off successively (on either directions) portions of arcs equivalent to the eighth part of a sign (30°). Join two and two of these marks by threads. Half of them (lengths of threads) will be the *R*sines."[10]

Mathematical Method.[11] In this method Brahmagupta employs the trigonometrical formulae

---

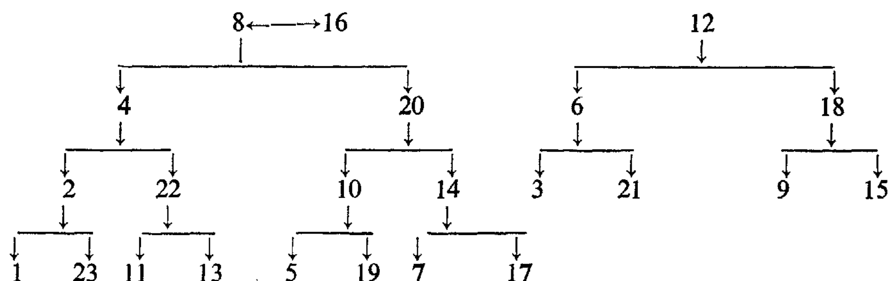
[1] *JIMS*, xv, (1924), pp. 121-6.
[2] *Ibid*, pp. 105-13 of "Notes and Questions."
[3] *PSi*, iv. 6-11, 12-15.
[4] *Ibid*, iv. 2-5.
[5] *ŚiDVṛ*, ii. 1-8.
[6] *ŚiDVṛ*, xiii. 2-3.
[7] *BrSpSi*, xxi. 16.
[8] Datta, Bibhutibhusan, "Hindu Values of π", *JASB*, N. S., Vol. 22 (1926), pp. 25-42; see particularly p. 32, footnote 1.
[9] His table will be found in *BrSpSi*, ii. 2-9.
[10] *BrSpSi*, xxi. 17.
[11] *Ibid*, xxi. 20-21; compare also the verse 23.

(*a*) $\sin \frac{\theta}{2} = \frac{1}{2}\sqrt{\sin^2 \theta + \text{versin}^2\, \theta}$

(*b*) $\sin (90^\circ - \frac{\theta}{2}) = \sqrt{1 - \sin^2 \frac{\theta}{2}}$

From the known value of the *R*sine of 8α, that is, of 30°, α being equal to $3^\circ\ 45'$, we can calculate, by (*a*), the *R*sines of 4α, 2α, α. Then by (*b*) will be obtained the *R*sine of 20α, 22α, 23α Again from the first two of the latter results, we shall obtain, by (*a*), the *R*sines of 10α and 11α; and thence by (*b*) the *R*sines of 14α and 13α. Continuing similar operations, we can compute the *R*sines of 5α and 19α, 7α and 17α. Again starting with the *R*sine of 12α, we shall obtain on proceeding in the same way, successively the values of the *R*sines of 6α and 18α; 3α and 21α; 9α and 15α. Thus the values of all the twenty-four *R*sines are computed.

It is perhaps noteworthy that *R*sin $n\alpha$ is called by Brahmagupta as the *n*th *R*sine. The successive order in which the various *R*sines have been obtained above can be exhibited as follows:



Brahmagupta then observes: "In this way (can be computed) the *R*sines in greater or smaller numbers, having known first the *R*sines of the sixth, fourth and third parts of the circumference of the circle."[1] He further remarks that the *R*sine of the semi-arc can be more easily calculated by the second formula of Varāhamihira.[2] Brahmagupta has also another table giving differences of *R*sines for every 15° of a circle of radius 150.[3]

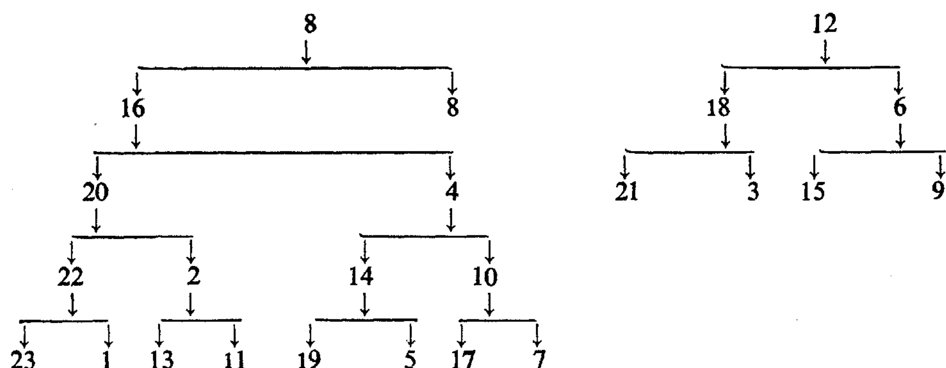
Āryabhaṭa II and Śrīpati

Āryabhaṭa II (950) gives the same table as that of the *Sūrya-siddhānta*.[4] But his method of computation is entirely different.[5] He takes recourse to the formulae

$$\sin \tfrac{1}{2}(90^\circ \pm \theta) = \sqrt{\tfrac{1}{2}(1 \pm \sin \theta)}$$

[1] *BrSpSi*, xxi. 22.
[2] *Ibid*, xxi. 23.
[3] *KK*, Part I, iii. 6; *DhGr*, 16.
[4] *MSi*, iii. 4-8.
[5] *MSi*, iii. 1-3.

Beginning with the known values of *R*sin 30° and *R*sin 45°, like Brahmagupta, the successive order in which the *R*sines will come out in the course of computation, can be best exhibited thus:



The table of Śrīpati (*c*. 1039) gives the *R*sines and versed *R*sines for every 3° 45′ of a circle of radius 3415.[1] His *first* method of computing it is the same as the graphic method of Brahmagupta. He says:

"Place marks at the eighth parts of a sign (30°); then (starting) from the joint of two quadrants, following up these marks, join two and two of them successively by means of threads; half of them will be the *R*sines."[2]

The *Second* method followed by Śrīpati is identical with the mathematical method of Brahmagupta.[3]

Bhāskara II

The table of Bhāskara II (1150) contains the *R*sines and versed *R*sines as well as their differences for every 3° 45′ of a circle of radius 3438′. He has indicated several methods of computing it. The *first* is practically the same as Brahmagupta's *graphic method*. He says:

"For computing the *R*sines, take any optional radius. On a plane ground describe a circle by means of a piece of thread equal to that radius. On it mark the cardinal points and 360 degrees; so in each quadrant of the circle there will be 90 degrees. Then divide every quadrant into as many equal parts as the number of *R*sines to be computed and put marks of these divisions. For instance, if it be required to calculate 24 *R*sines, there will be 24 marks. Then beginning from any of the cardinal points, and proceeding either ways, the threads connecting the successive points will be the chords. There will be thus 24 chords. Halves of these will be the *R*sines (required). So these half-chords should be measured and the results taken as the *R*sines."[4]

[1] *SiŚe*, iii. 3-10.
[2] *SiŚe*, xvi. 9.
[3] *SiŚe*, xvi. 14ff.
[4] *SiŚi*, *Gola*, v. 2-6 (Gloss).

The *second* is again a reproduction of Brahmagupta's *theoretical method:*

"When twenty-four *R*sines are required (to be computed), the *R*sine of 30° is the eighth element; its *R*cosine is the sixteenth; and *R*sin 45° is the twelfth. From these three elements, twenty-four elements can be computed in the way indicated. From the eighth we get the *R*sine of its half, that is, the fourth (element), its *R*cosine is the twentieth. Similarly from the fourth, the second and the twenty-second; from the second, the first and the twenty-third. In the same way from the eighth are obtained the tenth and fourteenth, fifth and nineteenth, seventh and seventeenth, eleventh and thirteenth. Again from the twelfth follow the sixth and eighteenth, third and twenty-first, ninth and fifteenth. The radius is the twenty-fourth *R*sine."[1]

The *third* method of computing trigonometrical tables described by Bhāskara II is the same as that of Āryabhaṭa II. The speciality of this method, as also of the two following, is, says Bhāskara II, that it does not employ the versed *R*sine function. As for the successive order of derivation, he points out that "from the eighth *R*sine (will be obtained) the sixteenth; from the sixteenth, the fourth and the twentieth; from the fourth, the tenth and fourteenth. In this way all the rest may be deduced."[2]

The *fourth* method of Bhāskara II is based on the application of the formula

$$R\sin \frac{1}{2}(\theta-\phi) = \frac{1}{2} \{(R\sin \theta - R\sin \phi)^2 + (R\cos \theta - R\cos \phi)^2 \}^{\frac{1}{2}},$$

"so that knowing any two *R*sines others may be derived. For instance, let one be the fourth *R*sine and the other eighth *R*sine. From them is derived the second *R*sine. From the second and fourth, the first; and so on."[3]

The *fifth* method depends on the formula

$$R\sin\ (45°-\theta)= \sqrt{\frac{1}{2}(R\cos \theta - R\sin \theta)^2}\ .$$

"Thus, for instance, take the eighth *R*sine; its *R*cosine is the sixteenth *R*sine. From these the fourth is derived; and so on."[4]

All the theoretical methods described above require the extraction of the square-root. So Bhāskara II propounds a new method (the *sixth*) in which that will not be necessary. It is based on the employment of the formula

$$R\cos 2\theta = R-\frac{2\ (R\sin\ \theta)^2}{R}$$

$$\text{or } \cos 2\theta=1-2 \sin^2 \theta.$$

[1] *SiŚi, Gola,* v. 2-6 (Gloss); xiv. 10-11 (Gloss).
[2] *SiŚi, Gola,* xiv. 12 (Gloss).
[3] *SiŚi, Gola,* xv. 13 (Gloss).
[4] *SiŚi, Gola,* xiv. 14 (Gloss).

But this method is defective in as much as "only certain elements of a table of *R*sines can be calculated thus,"[1] but not the whole table. This defect is present in a sense in the previous methods, for no one of the trigonometrical formulae employed in them suffices alone for the computation of a table containing more *R*sines (*vide infra*).

The *seventh* method of Bhāskara II for calculating a table of twenty-four *R*sines, has been described thus:

"Multiply the *R*cosine by 100 and divide by 1529; diminish the *R*sine by its $\frac{1}{467}$ part. The sum of these two results will give the next *R*sine and their difference the previous *R*sine. Here 225 less 1/7 is the first *R*sine. And by this rule can be successively calculated the twenty-four *R*sines."[2]

$$jy\bar{a}\,(n\alpha \pm \alpha) = \left(jy\bar{a}\,n\alpha - \frac{jy\bar{a}\,n\alpha}{467}\right) \pm \frac{100}{1529}\,kojy\bar{a}\,n\alpha,$$

where $n = 1, 2, \ldots, 24$; $\alpha = 3°\,45'$; and

$$jy\bar{a}\,\alpha = 225 - \frac{1}{7}.$$

The *rationale* of this formula is as follows:

By the Addition and Subtraction Theorems,

$$jy\bar{a}\,(n\alpha \pm \alpha) = \frac{1}{R}\,(jy\bar{a}\,n\alpha.\ kojy\bar{a}\,\alpha \pm kojy\bar{a}\,n\alpha.\ jy\bar{a}\,\alpha),$$

$$= jy\bar{a}\,n\alpha.\ \frac{kojya\,\alpha}{R} \pm kojy\bar{a}\,n\alpha.\ \frac{jy\bar{a}\,\alpha}{R}$$

Now

$$\frac{1}{R}\,jy\bar{a}\,\alpha = \frac{1}{3438}\left(225 - \frac{1}{7}\right) = \frac{787}{12033} = \frac{1}{15{\cdot}289707\ldots.}$$

$$= \frac{100}{1528{\cdot}9707\ldots} = \frac{100}{1529} \text{ nearly}$$

and $\frac{1}{R}\,kojy\bar{a}\,\alpha = \frac{1}{R}\sqrt{R^2 - (jy\bar{a}\,\alpha)^2} = \sqrt{1 - \left(\frac{jya\,\alpha}{R}\right)^2},$

$$= \sqrt{1 - \frac{1}{233{\cdot}775\ldots}} = 1 - \frac{1}{467{\cdot}550\ldots}$$

---

[1] *SiŚi, Gola*, xiv. 15 (Gloss).
[2] *SiŚi, Gola*, xiv. 18-20.

$$= 1 - \frac{1}{467} \text{ nearly}$$

and hence the rule. This formula is very nearly accurate. For according to the modern values

$$jy\bar{a}\ (3^\circ\ 45') = 224{\cdot}856\ldots.$$

$$\text{Therefore } \frac{1}{R}\ jy\bar{a}\ (3^\circ\ 45') = \frac{224.856}{3438} = \frac{1}{15{\cdot}28978\ldots.} = \frac{100}{15{\cdot}28978\ldots.}$$

Bhāskara II has indicated how to compute a table of *R*sines for every 3° of a circle of radius 3438′. He writes:

"For instance if (it be required to compute) thirty *R*sines in a quadrant, half the radius is the tenth *R*sine, its *R*cosine is the twentieth *R*sine. *R*sin 45° is the fifteenth *R*sine; *R*sin 36° is the twelfth and *R*cos 36° the eighteenth. The *R*sine of 18° is the sixth and its *R*cosine is the twenty-fourth. Then by the rule for deriving the *R*sine of the half arc from the square-root of the sum of the squares of the *R*sine and versed *R*sine of an arc, as stated before, from the tenth (is derived) the fifth; its *R*cosine is the twenty-fifth. In that way from the twelfth (is calculated) the sixth and twenty-fourth; from the sixth, the third and twenty-seventh; from the eighteenth, the ninth and twenty-first. These are the only elements (of the table) of *R*sines which can be calculated in this way. So it has been observed that 'only certain elements etc'. Next the formula for the *R*sine of half the difference of two arcs should be employed. Let the fifth be the one *R*sine and the ninth the other. From them will follow the second; its *R*cosine is the twenty-eighth *R*sine. From these two again by employing the (previous) rule for the *R*sine of semi-arcs from the square-root of the sum of the squares of the *R*sine and versed *R*sine, the first and fourteenth (are obtained). The remaining fourteen *R*sines can also be computed in the same way."[1]

Bhāskara II has further given a rule for computing a trigonometrical table for every degree. So it is called *Pratibhāgika-jyakā-vidhi* ("The rule for the *R*sine of every degree").

"Deduct from the *R*sine of any arc its 6567th part; multiply its *R*cosine by 10 and then divide by 573. The sum of these two results is the next *R*sine and their difference the preceding *R*sine. Here the first *R*sine (i.e. *R*sin 1°) will be 60′ and other *R*sines may be successively found. Thus in a circle of radius equal to 3438′, will be found 90 *R*sines."[2]

$$jy\bar{a}\ (\theta \pm 1^\circ) = \left(jy\bar{a}\ \theta - \frac{jy\bar{a}\ \theta}{6567}\right) \pm \frac{10}{573}\ kojy\bar{a}\ \theta$$

where $\theta = 1^\circ, 2^\circ, \ldots\ldots, 89^\circ$; given $jy\bar{a}\ 1^\circ = 60'$.

[1] *SiŚi, Gola*, xiv. 15 (Gloss).
[2] *SiŚi, Gola*, xiv. 16-8.

The *rationale* of this rule can be easily found : For by the Addition and Subtraction Theorems,

$$jy\bar{a}\,(\theta \pm 1^\circ) = \frac{1}{R}\,(jy\bar{a}\ \theta.\ kojy\bar{a}\ 1^\circ \pm kojy\bar{a}\ \theta.\ jy\bar{a}\ 1^\circ).$$

Now it is stated that $R = 3438'$ and $jy\bar{a}\ 1^\circ = 60'$. Therefore

$$\frac{1}{R}\,jy\bar{a}\ 1^\circ = \frac{60}{3438} = \frac{10}{573}$$

$$\frac{1}{R}\,kojy\bar{a}\ 1^\circ = \sqrt{1-(jya\ 1^\circ/R)^2} = \left\{1-\left(\frac{10}{573}\right)^2\right\}^{1/2}$$

$$= 1 - \frac{100}{2 \times 328329} = 1 - \frac{1}{6566{\cdot}58}$$

$$= 1 - \frac{1}{6567}$$

$$= {\cdot}999847723\ldots.$$

The denominator wrongly appears as 6569 in Bapu Deva's edition of the *Siddhānta-Śiromaṇi*.[1]

The short table of Bhāskara II contains differences of *R*sines for intervals of 10° in a circle of radius 120.[2]

Posterior Writers

Amongst the writers posterior to Bhāskara II (1150) who have given tables of trigonometrical functions, the most notable are Mahendra Sūri (1370) and Kamalākara (1658). The latter has a table of *R*sines and their differences for every degree of arc of a circle of radius 60, while the former gives tables of *R*sines and versed *R*sines together with their differences for every degree of the arc of a circle of radius 3600. Mahendra Sūri has furnished also some other tables for ready reckoning in Astronomy. It is noteworthy that Mahendra Sūri's work, *Yantrarāja*[3], is admittedly based upon some Arabic work. We are informed by his commentator Malayendu Sūri, a direct disciple of the author and who wrote his commentary only 12 years after the text, that the author was the court astrologer of some potentate of the name of Firoz, who is probably the famous Sultan Firoz Shah Tughluk of Delhi (1351-88 A.D.). The illustrations chosen will agree with this date.

[1]The text given by Bapu Deva runs as "*Svago'ṅgeṣusaḍaṁśena*....". It will be"*Svāgāṅgeṣuṣaḍaṃśena*...."

[2]*SiŚi, Graha*, ii. 13.

[3]Mss of the text with the commentary of *Yantrarāja* are available in the libraries of India Office, London (Nos. 2906-8), Benares Sanskrit College (No. 2905), Bikaner Palace (No. 760) and also at other places. Our copy has been procured from Benares.

The following Table gives the relevant details of the various *R*sine-Tables constructed in India from time to time.

Constructor of Table	Radius chosen	Interval taken	Sexagesimal places calculated
Author of *SūSi*[1]	3438′	225′	1 (minutes only)
Āryabhaṭa[2]	3438′	225′	1 (same Table as in *SūSi*)
Varāhamihira[3]	120	225′	2 (minutes and seconds)
Brahmagupta (1)[4]	3270	225′	1
(2)[5]	150	15°	1
Deva[6]	300	10°	1
Lalla (1)[7]	3438′	225′	1 (same Table as in Ā)
(2)[8]	150	10°	1
Sumati[9]	3438′	1°	1
Govinda Svāmi[10]	3437′44″19‴	225	3
Vaṭeśvara[11]	3437′44″	56′15″	2
Mañjula[12]	8°8′	30°	2 (degrees and minutes)
Āryabhaṭa II[13]	3438′	225′	1
Srīpati[14]	3415′	225′	1
Udayadivākara[15]	12375859‴ or 3437′44″19‴	225′	1 (thirds only)
Bhāskara II[16] (1)	3438′	225′	1
(2)	120	10°	1
Brahmadeva[17]	120	15°	1
Vṛddha Vaśiṣṭha[18]	1000	10°	1
Malayendu Sūri[19]	3600	1°	2
Madanapāla[20]	21600	1°	2
Mādhava[21]	3437′ 44″ 48‴	225′	3
Parameśvara[22]	3437′ 44″	225′	2
Munīśvara[23]	191	1°	4
Kṛṣṇa-daivajña[24]	500	3°	1
Kamalākara[25]	60	1°	5
Jagannātha Samrāṭa[26]	60	30′	5

[1] *SūSi*, ii. 17-22(a-b).
[2] *Ā*, i. 12.
[3] *PSi*, iv. 6-12.
[4] *BrSpSi*, ii. 2-5.
[5] *KK*, Part 1, iii. 6; *DhGr*, 16.
[6] *KR*. i. 23.
[7] *ŚiDVr*, I, ii. 1-8.
[8] *Ibid*, xiii. 3.
[9] *SMT* and *SK*.
[10] His com. on *MBh*, iv. 22.
[11] *VSi*, II, i. 2-26.
[12] *LMā*, ii. 2(c-d).
[13] *MSi*, iii. 4-6(a-b).
[14] *SiŚe*, iii. 3-6.
[15] His com. on *LBh*, ii. 2-3.
[16] *SiŚi*, *Gaṇita*, ii. 3-6; 13.
[17] *KPr*, ii. 1.
[18] *VVSi*, ii. 10-11.
[19] *YR*, i. 5, commentary.
[20] His com. on *SūSi* xii. 83.
[21] See Nīlakaṇṭha's com. on *Ā*, ii. 12.
[22] His com. on *LBh*, ii. 2(c-d)-3(a-b).
[23] *SiSā*, ii. 3-18.
[24] *KKau*, ii. 4-5.
[25] *SiTVi*, ii. pp. 244-5 (Lucknow Edition).
[26] *Siddhānta-samrāṭ*, ii. beginning.

6. Interpolation

Function of any arc

For finding the trigonometrical functions of an arc, other than those whose values have been tabulated, the Hindus generally follow the principle of proportional increase. Thus the *Sūrya-siddhānta* says:

"Divide the minutes (into which the given arc is first reduced) by 225; the quotient will indicate the number of tabular *R*sines exceeded; (the remainder) is multiplied by the difference between the (tabular) *R*sine exceeded and that which is still to be reached and then divide by 225. The result thus obtained should be added to the exceeded tabular *R*sine; (the sum) will be the (required) direct *R*sine. This rule is applicable also to the case of (determining) the versed *R*sine."[1]

Brahmagupta states:

"Divide the minutes by 225, the quotient (will indicate) the number of tabular *R*sines (exceeded); the remainder is multiplied by the (next) difference of *R*sines and divided by the square of 15; the result is added to the (tabular) *R*sine coresponding to the quotient. Such (is the method) for finding the *R*sine."[2]

Such rules appear also in other astronomical works.[3]

The method of Mañjula (932) is very simple though it yields results only roughly approximate. He says:

"The sum of the signes (in the given arc successively) multiplied by 4, 3 and 1 will give the degrees in the *R*sines and *R*cosines (to be found); such are the minutes."[4]

This rule though it appears to be cryptic has been fully explained by the commentators; Praśastidhara (968), Parameśvara (1430) and Yallaya (1482). We shall explain it with the help of a simple illustrative example: Suppose it is necessary to find the *R*sine of the angle 76° 30′. This angle can be written as 2 signs 16° 30′. Now the rule says that for the first sign take 4 and for the second sign 3. For the third sign of 30° we are to take 1, so for a portion 16° 30′ of that sign we should take $(16\frac{1}{2}\times 1)/30$. The sum of these 4, 3 and 33/60 will be the degrees in the required *R*sine; and they will also be the minutes of the required *R*sine. Therefore

[1] *SūSi*, ii. 31-2.
[2] *BrSpSi*, ii. 10; compare also *KK*, Part I, iii. 6.
[3] *MBh*, iv. 3-4; *LBh*, ii. 2-3;, *ŚiDVr*, ii. 12; *MSi*, iii. 10½; *SiŚi*, *Graha*, ii. 10½; *SiTVi* ii. 171; *SiŚe*, iii. 15.
[4] "*Catustrekaghnarāśyaikyaṁ doḥkoṭyoraṁśakāḥ kalāḥ.*"—*LMā*, ii. 2.

Jyā (76° 30′) =(4+3+33/60) degrees+(4+3+33/60) minutes
=7° 40′ 33″

The *rationale* of this rule which has also been given by the earlier commentators, is this: Mañjula considers the circle of reference to be of radius 488′ or in sexagesimal notation 8° 8′; and his Table is very short:

Arcs	*R*sines	Differences
0°	0° 0′	
		4° 4′
30°	4° 4′	
		3° 3′
60°	7° 7′[1]	
		1° 1′
90°	8° 8′	

To find the *R*sine of any intermediate arc he applies the principle of proportional increase. And this at once leads to the rule.

Arcs of Functions

The Hindus employed the principle of proportional increase also for the inverse problem of finding the arc which has a given trigonometrical function different from those tabulated. The *Sūrya-siddhānta* says:

"Subtract the (nearest smaller tabular) *R*sine (from the given *R*sine) multiply the remainder by 225 and divide by that difference (i.e. the difference corresponding to the interval in which the given *R*sine lies); the quotient added to the number (corresponding to the tabular *R*sine subtracted) multiplied by 225 will give the arc (required)."[2]

Brahmagupta writes:

"Subtract the (nearest smaller tabular) *R*sine (from the given *R*sine); the remainder is multiplied by 225 and divided by the (tabular) difference of *R*sines; the

[1]Accurately speaking
jyā 60° $= 488 \times \frac{\sqrt{3}}{2} =$ 7° 2′.608....;
Mañjula takes the value to be 7° 7′ obviously with the purpose of simplifying his rule.
[2]*SūSi*, ii. 33.

result should then be added to the product of the number corresponding to the subtracted *R*sine and the square of 15: (this will be) the arc (required)."[1]

Similarly in other works.[2]

Second Difference

The process explained above for calculating the trigonometrical functions of a given arc or the arc having given trigonometrical functions, will yield results correct only to a first degree of approximation in as much as the first difference alone of the tabular *R*sines has been employed.[3] More accurate results will be obtained by taking into consideration also the second (and higher) differences. The earliest Hindu writer to do so was Brahmagupta. It is perhaps noteworthy that this more correct method of interpolation does not occur in his bigger work, *Brāhma-sphuṭa-siddhānta*, which was composed in 628 A.D. but in his earlier monograph *Dhyānagrahopadeśa* as well as in his later work *Khaṇḍa-Khādyaka* written in 665 A.D. These latter works, as has been stated before, contain a table of differences of *R*sines for every arc of 15° in a circle of radius equal to 150. He says:

"Half the difference between the (tabular) difference passed over and that to be passed is multiplied by (residual) minutes and divided by 900; half the sum of those differences plus or minus that quotient according as it is less or greater than the (tabular) difference to be passed, will be the (corrected) value of the difference to be passed over."[4]

Suppose it is required to calculate the function—*R*sine, *R*cosine or versed *R*sine—of an arc α'. Let α_1, α_2, α_3 be the three consecutive values of the argument in the table such that $\alpha_3 > \alpha_2 > \alpha_1$.

Values of the argument α	Values of the function $f(\alpha)$	Differences of functions
α_1	f_1	
		Δ_1
α_2	f_2	
		Δ_2
α_3	f_3	

[1] *BrSpSi*, ii. 11; compare also *KK*, iii. 12; *DhGr*, 70.

[2] *MBh*, viii. 6; *ŚiDVṛ*, ii. 13; *MSi*, iii. 12; *SiŚe*, iii. 16; *SiŚi*, *Graha*, ii. 11f; *SiTVi*, ii. 172-3.

[3] The roughness of the result is due also to other causes. Bhāskara II observes: "As much large the radius of the circle is and into as many large number of (equal) arcs its quadrant is divided, so much accurate will be the *R*sines (calculated). Otherwise (the result) will be rough (*sthūla*)". ——*SiŚi*, *Graha*, ii. 15 (*Gloss*).

[4] "*Gatabhogyakhaṇḍakāntaradalavikalavadhācchatairnavabhirāptyā*
Tadyutidalaṁ yutonaṁ bhogyādūnādhikaṁ bhogyam" *KK*, Part 2, i,4; *DhGr*, 17.
See Sengupta, P. C., "Brahmagupta on Interpolation," *BCMS*, xxii, 1931.

Now if $\alpha_3 > \alpha' > \alpha_2$ for the calculation of $f(\alpha')$, f_2 will be technically called "the function exceeded," f_3 "the function to be reached," Δ_1 "the difference passed over" and Δ_2 "the difference to be passed." Let $\alpha' - \alpha_2 = r$.

Now suppose that $\alpha_3 - \alpha_2 = \alpha_2 - \alpha_1 = h$, say. Then according to the rules stated by all Hindu astronomers,

$$f(\alpha') = f_2 + \frac{r}{h}\,\Delta_2;$$

which is correct up to the first order of approximation. To get more accurate results, says Brahmagupta

$$\frac{\Delta_1+\Delta_2}{2} \pm \frac{r}{h}\left(\frac{\Delta_1 \sim \Delta_2}{2}\right)$$

should be taken as the value of "the difference to be passed", instead of Δ_2; the positive or negative sign being taken, according as

$$\frac{\Delta_1+\Delta_2}{2} < \text{or} > \Delta_2\,.$$

Therefore, according to the method of Brahmagupta

$$f(\alpha') = f_2 + \frac{r}{h}\left\{\frac{\Delta_1+\Delta_2}{2} \pm \frac{r}{h}\left(\frac{\Delta_1 \sim \Delta_2}{2}\right)\right\}.$$

In the rule h is stated to be 900, as it was formulated with a view at the table of the *Khaṇḍa-Khādyaka*, in which the interval between the consecutive values of the argument, is 15° or 900′. This equation can be written in the form

$$f(\alpha') = f_2 + \frac{r}{h}\,\Delta_2 + \frac{r}{2h}\left(1 \pm \frac{r}{h}\right)(\Delta_1 \sim \Delta_2);$$

which agrees with the formula method of interpolation, correct up to the second degree.[1]

[1]It may be mentioned here that the formula

$$f(\alpha') = f_2 + \frac{r}{h}\,\Delta_1 - \frac{r}{2h}\left(1 + \frac{r}{h}\right)(\Delta_1 - \Delta_2)$$

has been stated by Vaṭeśara (904) in his *Siddhānta* (Ch. 2, sec. 1, vss. 65-6) and the formula

$$f(\alpha') = f_2 + \frac{r}{h}\,\Delta_2 + \frac{r}{2h}\left(1 - \frac{r}{h}\right)(\Delta_1 - \Delta_2)$$

by Govindasvāmi (8th century) in his com. on *Mahā Bhāskarīya* (iv. 22) and by Parameśvara (1408) in his com. on *Laghu Bhāskarīya* [ii. 2(c-d)-3(a-b)]. Govindasvāmi has, however, prescribed it for the second sign only.

As has been observed by Bhāskara *II*[1] in the above formula one has to take the negative sign in calculating the *R*sine functions and the positive sign for the versed *R*sines. For in case of *R*sine functions, the first difference continuously decreases as the argument increases, while contrary is the case with the versed *R*sine functions. Therefore the mean value of any two differences $\left(\text{i.e. } \frac{\triangle_1 + \triangle_2}{2}\right)$ is greater than the succeeding one ($\triangle_2$) in case of *R*sine functions and less in case of versed *R*sine functions.

Brahmagupta's method of interpolation appears also in the works of Mañjula (932) thus:

"(Find) half the sum of the tabular difference passed over and that to be passed; half their difference is multiplied by the (remaining) degrees etc. and divided by 30; half the sum minus this quotient will be the corrected value of the difference of (tabular) *R*sines to be passed in the (calculations of the *Laghu*)—*Mānasa*."[2]

The divisor is stated to be 30 in this rule, as Mañjula's table of *R*sines contains values at intervals of 30° each.

Bhāskara II (1150) writes:

"The difference of the (tabular) difference passed over and that to be passed is multiplied by the remaining degrees and divided by 20; half the sum of the (tabular) difference passed over and that to be passed minus or plus that quotient will be the corrected value of the difference to be passed over in calculation here for *R*sines and versed *R*sines."[3]

In formulating this rule Bhāskara II had in view a table calculated at intervals of 10°. The *rationale* of the rule has been explained by him thus:

[1]"*....ūnaṁ kriyate yataḥ kramajyākaraṇe khaṇḍānyapacayena vartante. Utkramajyākaraṇe tūpacayenātastatra yutamityupapannam.*"—*SiŚi, Graha*, ii. 16 (Gloss).

[2]"*Gataiṣyakhaṇḍayogārdhamantarārdhena saṁguṇāt*
Bhāgādeḥ khāgnilabdhonaṁ bhogyajyā Mānase sphuṭāḥ."

There is a bit of uncertainty about the authenticity of this verse. In the Calcutta University Collection, there are three manuscripts of the *Laghu-Mānasa* and four commentaries which contain also the text. The commentary of Praśastidhara (958) is "copied from Ms. No. B 583 and compared with other Mss in the Oriental Library, Mysore." That by Parameśvara (1430) is "copied from a palm leaf manuscript in Malayalam character belonging to the office of the curator for the publication of Sanskrit Mss., Trivandrum." The source of the commentary of Yallaya (1482) is not mentioned. The above verse appears in the commentaries of Praśastidhara and Yallaya. But in the latter it has been attributed to Mallikārjuna-sūri. Now this writer flourished about 1180 A.D. Thus he is posterior to Praśastidhara by more than two centuries. So it is not possible for the latter to borrow anything from the former. I think the mistake has been made by Yallaya. The verse in question seems to me to be due in fact to Mañjula, and is more particularly from his *Bṛhat-mānasa*, which is now lost. Praśastidhara has quoted copiously from that work in his commentary of the *Laghu-mānasa* without, however, expressly mentioning it.

[3]*SiŚi, Graha*, ii.16.

"Half the sum of the (tabular) difference passed over and that to be passed will be the difference at the middle of those differences. But the difference to be passed is at the end of that interval to be passed. Hence proportion (should be taken) with their difference: If for an interval of 10°, we obtain half the difference of them, then what will be obtained for (an interval of) the remaining degrees? Thus by the rule of three, 20 will be the divisor of the product of the remaining degrees and the difference of the (tabular) difference passed over and that to be passed. By the quotient is then diminished half the sum of the (tabular) difference passed over and that to be passed; for in the calculations of *R*sines the differences are in the decreasing order. But in the calculations of versed *R*sines they are in the increasing order and hence the plus in this case. Thus (the rule) is proved."

This method of interpolation has been severely criticised by Kamalākara.[1] But he is wrong. Munīśvara (1646) attempted to modify this method by iterating the process but his process of iteration is in correct as he has replaced the (tabular) difference to be passed by the instantaneous difference, at every stage.[2]

The *rationale* of the rule can be shown with the help of trigonometry to be as follows:

$$\triangle_1 = \sin \alpha_2 - \sin \alpha_1 = \sin \alpha_2 - \sin (\alpha_2 - h) = \sin \alpha_2 (1 - \cos h) + \cos \alpha_2 \sin h).$$

Since h is small we can expand $\cos h$ and $\sin h$ in powers of h; then neglecting powers higher than the second, we get

$$\triangle_1 = h \cos \alpha_2 + \frac{h^2}{2} \sin \alpha_2 .$$

Similarly, $\triangle_2 = h \cos \alpha_2 - \frac{h^2}{2} \sin \alpha_2 .$

Therefore, $\frac{\triangle_1 + \triangle_2}{2} = h \cos \alpha_2$ and $\frac{\triangle_1 - \triangle_2}{2} = \frac{h^2}{2} \sin \alpha_2 .$

Now, if $\alpha' = \alpha_2 + r$, up to the second order of approximation, we have

$$\sin \alpha' = \sin (\alpha_2 + r) = \sin \alpha_2 \left(1 - \frac{r^2}{2} \right) + r \cos \alpha_2 ,$$

Therefore,

$$\sin \alpha' = \sin \alpha_2 + \frac{r}{h} \left(\frac{\triangle_1 + \triangle_2}{2} - \frac{r}{h} \frac{\triangle_1 - \triangle_2}{2} \right) .$$

[1] *SiTVi*, ii. 175-83.

[2] For Munīśvara's process of iteration, see Gupta R. C., "Munīśvara's modification of Brahmagupta's rule for second order interpolation," *IJHS*, vol. 14, no. 1, 1979, pp. 66-72.

Evidently in this case ,

$$\frac{\Delta_1+\Delta_2}{2} > \Delta_2$$

Hence, $\sin \alpha' = \sin \alpha_2 + \frac{r}{h}\left(\frac{\Delta_1+\Delta_2}{2} - \frac{r}{h}\frac{\Delta_1 \sim \Delta_2}{2}\right)$

In case of versin functions

$\Delta_1 = \text{versin}\, \alpha_2 - \text{versin}\, (\alpha_2 - h)$,

$= \cos(\alpha_2 - h) - \cos \alpha_2 = h \sin \alpha_2 - \frac{h^2}{2} \cos \alpha_2$;

$\Delta_2 = \text{versin}\, (\alpha_2 + h) - \text{versin}\, \alpha_2$,

$= \cos \alpha_2 - \cos(\alpha_2 + h) = h \sin \alpha_2 + \frac{h^2}{2} \cos \alpha_2$.

Therefore $\qquad \frac{\Delta_1+\Delta_2}{2} < \Delta_2$

and $\frac{\Delta_1+\Delta_2}{2} = h \sin \alpha_2$, $\frac{\Delta_1-\Delta_2}{2} = -\frac{h^2}{2} \cos \alpha_2$.

Now $\text{versin}\, \alpha' = 1 - \cos \alpha' = 1 - \cos(\alpha_2 + r)$,

$= \text{versin}\, \alpha_2 + r \sin \alpha_2 + \frac{r^2}{2} \cos \alpha_2$.

Therefore,

$\text{versin}\, \alpha' = \text{versin}\, \alpha_2 + \frac{r}{h}\left(\frac{\Delta_1+\Delta_2}{2} + \frac{r}{h} \cdot \frac{\Delta_1 \sim \Delta_2}{2}\right)$.

Combining these two results we get Brahmagupta's formula.

This method of interpolation has been applied also to the inverse problem of finding the arc having a given trigonometrical function.

Brahmagupta says:

"To find the arc, multiply the residue (after subtracting as many *R*sines from the given quantity as possible) by 900 and divide by the difference to be passed after having determined that difference by repeated operations. The degrees in the quotient will be the arc of the residue. Subtract (as many possible) *R*sines (from the given quantity), multiply the residue by 900 and divide by the (next) difference not subtracted; the quotient will be the second residue; multiply it by half the difference between the

(tabular) difference passed over and that to be passed and then divide by 900. With this quotient proceed as before for the (adjusted) value of the (tabular) difference to be passed. Repeat the same operations with the residue until the result is obtained finally."[1] (By "*R*sines" in this rule is meant "tabular *R*sine-differences.")

The latter portion of this rule has become rather cryptic, as all the successive operations have not been fully described. But it appears from the explanations of the commentator Bhaṭṭotpala (966) that Brahmagupta has intended the same formula as has been clearly described by Bhāskara II. The latter says:

"Subtract the (tabular) differences (as many as possible from the given value); multiply half the remainder by the difference of the (tabular) difference passed over and that to be passed, and then divide by the (tabular) difference to be passed. Half the sum of the (tabular) difference passed over and that to be passed plus or minus the quotient is the adjusted value of the (tabular) difference to be passed, whence (will follow) the arc (required)."[2]

He then remarks as before that the negative sign should be taken in calculating the *R*sines and the positive sign for the versed *R*sines.

The *rationale* of this will be clear from the previous formula on inversion. There we shall now have $f(\alpha')$ known and $r\ (=\alpha'-\alpha_2)$ as unknown.

$$f=f_2+\frac{r}{h}\left(\frac{\Delta_1+\Delta_2}{2}\pm\frac{r}{h}\,\frac{\Delta_1-\Delta_2}{2}\right)$$

or $$\frac{r}{h}\,\Delta_2=f-f_2+\frac{r}{h}\,\frac{\Delta_2-\Delta_1}{2}\mp\frac{r^2}{h^2}\,\frac{\Delta_1-\Delta_2}{2}.$$

Now let us take for the first approximation, as before

$$\frac{r}{h}=\frac{f-f_2}{\Delta_2}.$$

Substituting this value of $\frac{r}{h}$ in the neglected terms; we get for the second approximation

$$r=(f-f_2)\,\frac{h}{\Delta_2}\left(1+\frac{\Delta_2-\Delta_1}{2\Delta_2}\mp\frac{f-f_2}{\Delta_2^2}\cdot\frac{\Delta_1-\Delta_2}{2}\right)$$

$$=(f-f_2)\,\frac{h}{\Delta}\text{, say}$$

[1] *KK*, Part 2, i. 12. The printed text gives only the earlier part of the rule. The remaining portion has been taken from the text of Bhaṭṭotpala.
[2] *SiŚi Graha*, ii. 17.

so that $\triangle$ will be the adjusted value of the (tabular) difference to be passed. Then

$$\frac{1}{\triangle} = \frac{1}{\triangle_2}\left(1 + \frac{\triangle_2 - \triangle_1}{2\triangle_2} \mp \frac{f - f_2}{\triangle_2^2}\,\frac{\triangle_1 - \triangle_2}{2}\right)$$

Therefore, $$\triangle = \triangle_2\left(1 + \frac{\triangle_2 - \triangle_1}{2\triangle_2} \mp \frac{f - f_2}{\triangle_2^2}\,\frac{\triangle_1 - \triangle_2}{2}\right)^{-1}$$

$$= \triangle_2\left(1 - \frac{\triangle_2 - \triangle_1}{2\triangle_2} \pm \frac{f - f_2}{\triangle_2^2}\,\frac{\triangle_1 - \triangle_2}{2}\right)$$

or $$\triangle = \frac{\triangle_1 + \triangle_2}{2} \pm \frac{f_1 - f_2}{2\triangle_2}(\triangle_1 - \triangle_2),$$

as stated in the rule. But the more accurate result by inversion would have been

$$\triangle = \frac{\triangle_1 + \triangle_2}{2} \pm \frac{f - f_2}{\triangle_1 + \triangle_2}(\triangle_1 - \triangle_2).$$

Bhāskara II was clearly aware of this. For he is found to have remarked:

"This improved formula for calculating the arc is a little rough (*sthūla*). Though rough, it has been adopted for its simplicity (*sukhārtha*). By other means such as finer calculations or repeated applications it can be made more accurate."[1]

Generalised Formula

Brahmagupta has extended his formula of interpolation so as to be applicable also to the case when the intervals between the consecutive tabular values of the argument are not equal. He says:

"Multiply the increase of the *Śīghra* anomaly to be passed by the degrees of the increase of the *Śīghra* equation passed over and divide by the increase of the *Śīghra* anomaly passed over; the quotient is the (adjusted) increase of the *Śīghra* equation in degrees. Multiply half the difference of that and the increase of the *Śīghra* equation to be passed by the residue of the anomaly and divide by the increase of the *Śīghra* anomaly to be passed; half the sum of these equations is decreased or increased by the (last) quotient, according as it (half the sum) is greater or less than the *Śīghra* equation to be passed; the result will be corrected *Śīghra* equation to be passed."[2]

---

[1]"*Idam dhanuḥkhaṇḍasphuṭīkaraṇaṁ kiñcit sthūlam. Sthūlamapi sukhārthamaṅgīkṛtam. Anyathā bījakarmaṇā vā' sphuṭaṁ kartuṁ yujyate*"—*Ibid* (Gloss)."

[2]*Bhuktagatiphalāṁśaguṇā bhogyagatirbhuktagatihṛtā labdhaṃ*
Bhuktagateḥ phalabhāgāstadbhogyaphalāntarārdhahataṁ
Vikalaṁ bhogyagatihṛtaṁ labdhenonādhikaṁ phalaikyārdhaṁ
Bhogyaphalādadhikonaṁ tadbhogyaphalaṁ sphuṭaṁ bhavati"—*KK*, Part 2, ii. 2-3.

That is, if $\alpha_3-\alpha_2 \neq \alpha_2-\alpha_1$, let $\alpha_3-\alpha_2=h_2$ and $\alpha_2-\alpha_1=h_1$.

Then

$$f(\alpha')=f_2+\frac{r}{h_2}\left\{\left(\frac{1}{2}\,\frac{\triangle_1\times h_2}{h_1}+\triangle_2\right)\pm\frac{r}{h_2}\cdot\frac{1}{2}\left(\frac{\triangle_1\times h_2}{h_1}-\triangle_2\right)\right\},$$

or $$f(\alpha')=f_2+\frac{r}{h_2}\,\triangle_2+\frac{r}{2h_2}\left(1\pm\frac{r}{h_2}\right)\left(\frac{\triangle_1\times h_2}{h_1}-\triangle_2\right),$$

where the upper or lower sign is to be taken according as

$$\frac{1}{2}\left(\frac{\triangle_1\times h_2}{h_1}+\triangle_2\right)< \text{ or } >\triangle_2.$$

7. Spherical Trigonometry

Solution of Spherical Triangles

From the use in the treatment of astronomical problems, we find that the Hindus knew how to solve spherical triangles, of oblique as well as of right-angled varieties. They do not seem to possess a general method of solution in this matter unlike the Greeks who systematically followed the method of Ptolemy (*c*. 150) based on the well-known theorem of Menelaus (90). Still with the help of the properties of similar plane triangles and of the theorem of the square of the hypotenuse, they arrived at a set of accurate formulae sufficient for the purpose. As has been proved conclusively by Sengupta[1], Braunmühl was wrong in supposing that in the matter of solution of spherical triangles the Hindus utilised the method of projection contained in the Analemma of Ptolemy.

Right-angled Spherical Triangle

The Hindus obtained the following formulae for the right-angled spherical triangle, right-angled at C:

(*i*) $\sin a=\sin c \sin A$,
(*ii*) $\cos c=\cos a \cos b$,
(*iii*) $\sin c \cos A=\cos a \sin b$,
(*iv*) $\sin b=\tan a \cot A$,
(*v*) $\cos A=\tan b \cot c$.

It should be particularly noted that as the tangent and cotangent functions are

[1]Sengupta, P. C., "Greek and Hindu Methods in Spherical Trigonometry", *Journ. Dept. Letters. Cal. Univ.*, Vol. xxi (1931).

[2]Braunmühl, *Geschichte*, pp. 38ff; compare also Heath, *Greek Math.*, II, p. 291.

not recognised in Hindu Trigonometry, the formulae (*iv*) and (*v*) are ordinarily written in the forms

$$(iv)\ \sin b = \frac{\sin a.\ \cos A}{\cos a.\ \sin A}, \qquad (v)\ \cos A = \frac{\sin b.\ \cos c}{\cos b.\ \sin c},$$

These formulae were obtained thus: (See Fig. 13)

Let ABC be a spherical triangle right-angled at C, lying on a sphere whose centre is at O. Produce the sides AC and AB to P and Q respectively, such that the arc AP=arc AQ=90°. Join PQ by an arc of the great circle on the sphere. Produce the arcs CB and PQ so as to meet at the point Z.

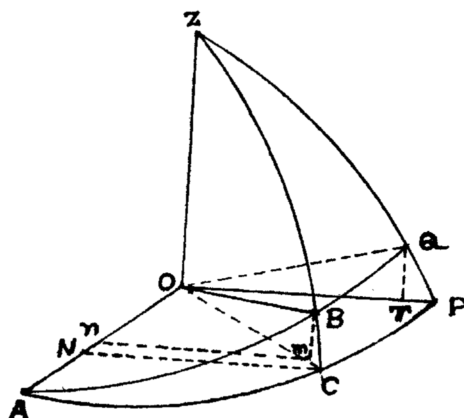


Fig. 13

Then clearly Z is the pole of the circle ACP as A is of the circle PQZ. Join A, B, C, P, Q, Z, with O. Draw QT perpendicular to OP, Bm to OC, Bn and CN to OA. Join mn.

From the similar triangles OQT and nBm

$$\frac{nB}{OQ} = \frac{nm}{OT} = \frac{Bm}{QT} \tag{1}$$

and from the similar triangles Onm and ONC

$$\frac{On}{ON} = \frac{Om}{OC} = \frac{nm}{NC}. \tag{2}$$

Substituting the values in terms of trigonometrical functions, we get

$$\frac{jy\bar{a}\ c}{R} = \frac{nm}{kojy\bar{a}\ A} = \frac{jy\bar{a}\ a}{jy\bar{a}\ A} \tag{1.1}$$

$$\frac{kojy\bar{a}\ c}{kojy\bar{a}\ b} = \frac{kojy\bar{a}\ a}{R} = \frac{nm}{jy\bar{a}\ b}. \tag{2.1}$$

Hence from (1.1), we get

$$jy\bar{a}\ a = \frac{jy\bar{a}\ c.\ jy\bar{a}\ A}{R},$$

which is of course equivalent to

$$\sin a = \sin c \sin A. \tag{3.i}$$

Similarly from (2.1)

$$kojy\bar{a}\ c = \frac{kojy\bar{a}\ a.\ kojy\bar{a}\ b,}{R}$$

or $$\cos c = \cos a \cos b. \tag{3ii}$$

Again equating the values of *mn* from (1.1) and (2.1), we get

$$jy\bar{a}\ c.\ kojy\bar{a}\ A = kojy\bar{a}\ a.\ jy\bar{a}\ b;$$

that is, $$\sin c \cos A = \cos a \sin b, \tag{3.iii}$$

Eliminating $\sin c$ between (3.i) and (3.iii), we get

$$\sin b = \tan a \cot A; \tag{3.iv}$$

and eliminating $\cos a$ between (3.ii) and (3.iii), we have

$$\cos A = \tan b \cot c. \tag{3.v}$$

As an illustration of the application of the above formulae let us take the problem of determination of the Sun's right ascension (α) when the Sun's longitude (λ) and declination (δ) are known. Let γM be the equator, γS the ecliptic and S the Sun. Then $\lambda (= \gamma S)$ denotes the sun's longitude, δ ($=SM$) the Sun's declination, and $\alpha (=\gamma M)$ the Sun's right ascension. If ϵ denotes the obliquity of the ecliptic, then by the formula (iii), we get

$$\sin \alpha = \frac{\sin \lambda.\cos \epsilon}{\cos \delta}$$

This result is stated by Āryabhaṭa[1], Brahmagupta[2] and others[3]. It is noteworthy that in this particular case the triangles *Bnm* and *OQT* are called technically *Krānti-Kṣetra* or "declination triangles" which shows definitely that they were actually drawn and the final result was actually obtained by the method stated above.[4]



Fig. 14

Oblique Spherical Triangle

For the solution of an oblique spherical triangle the Hindus had equivalents of the following formulae:

(*i*) $\cos a = \cos b \ \cos c + \sin b \sin c \ \cos A,$

(*ii*) $\sin a \ \cos C = \cos c \ \sin b - \sin c \ \cos b \ \cos A,$

(*iii*) $\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$

Let ABC be a spherical triangle lying on a sphere whose centre is at O. Produce the arc AC to S and arc CB to Q, such that the arc CS = arc $CQ = 90°$. Then C will be the pole of the great circle SQN. Join OA, OS and ON. Through B draw the small circle RVR' perpendicular to OA, intersecting the great circle SQN at D and D'. Join DD' intersecting SON at V. Let the diameter RVR' of the small circle cut OA at O'. Again through O draw the straight line WOE parallel to DVD'. Draw the great circle $KEK'W$ parallel to the small circle RBR' and the great circle EAW perpendicular to the latter and cutting it at F and F'. From B draw BT perpendicular to RVR', BH to DVD' and BM to OQ. Join MH cutting WOE at L. Draw NY perpendicular to OA, QQ' and MM' to OS. Let BH cut FF' at G.

From the similar triangles BMH and OYN, we get

$$\frac{BM}{OY} = \frac{MH}{NY} = \frac{HB}{NO}\ ; \tag{1}$$

and from the similar triangles OVO' and ONY

$$\frac{OV}{ON} = \frac{VO'}{NY} = \frac{O'O}{YO}\ . \tag{2}$$

---

[1] *Ā*, iv. 25.
[2] *BrSpSi*, iii. 15.
[3] *SūSi*, iii. 41-43; *PS*, iv. 29; *SiŚi*, *Graha*, ii. 54-5 etc.
[4] Compare Sen Gupta, P.C., "*Papers on Hindu Mathematics and Astronomy*", Part I, Calcutta, 1916, pp. 46ff; *PS*, iv. 35 (comments).
[5] Braunmühl, *Geschichte*, I, p. 41; Kaye, G.R., "Ancient Hindu Spherical Astronomy", *JASB*, Vol. xv (1919), pp. 153ff; P. C. Sen Gupta, *Papers etc.*, pp. 57ff.

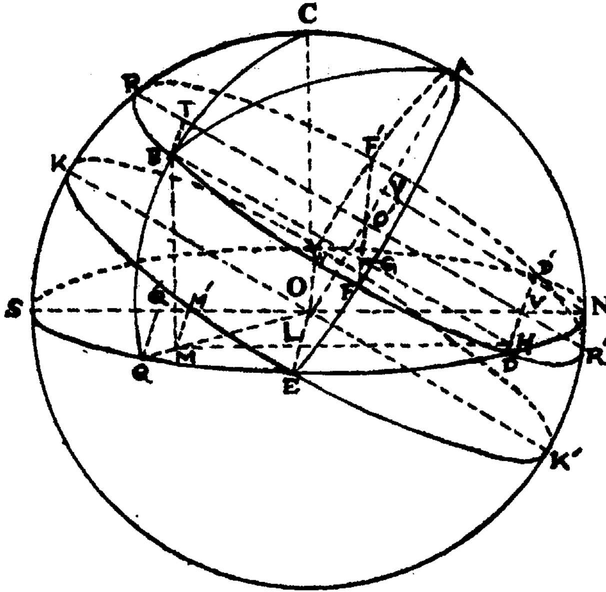


Fig. 15

Hence substituting the values

$$\frac{kojy\bar{a}\ a}{kojy\bar{a}\ (90°—b)} = \frac{MH}{jy\bar{a}\ (90°—b)} = \frac{HB}{R}$$

and $$\frac{OV}{R} = \frac{VO'}{jy\bar{a}\ (90° — b)} = \frac{kojy\bar{a}\ c}{kojy\bar{a}\ (90° — b)}\ ;$$

whence $$HB = \frac{R\ kojy\bar{a}\ a}{jy\bar{a}\ b},\quad MH = \frac{kojy\bar{a}\ a.\ kojy\bar{a}\ b}{jy\bar{a}\ b}, \qquad (1.1)$$

and $$OV = \frac{R\ kojy\bar{a}\ c}{jy\bar{a}\ b},\quad O'V = \frac{kojy\bar{a}\ b.\ kojy\bar{a}\ c}{jy\bar{a}\ b} \qquad (2.1)$$

Further $$\left.\begin{aligned} O'R &= jy\bar{a}\ c, \\ RT &= \frac{jy\bar{a}\ c\ utjy\bar{a}\ A}{R}, \\ ML &= \frac{jy\bar{a}\ \bar{a}.\ kojy\bar{a}\ (C - 90°)}{R} \end{aligned}\right\} \qquad (3)$$

Now $HB = VT = RO' + O'V - RT$.

Therefore substituting the values of the constituent elements on either sides of equations from (1.1), (2.1) and (3), we get

$$\frac{R \text{ kojyā } a}{\text{jyā } b} = \text{jyā } c + \frac{\text{kojyā } b.\ \text{kojyā } c}{\text{jyā } b} - \frac{\text{jyā } c\ .\ \text{utjyā } A}{R},$$

$$= \frac{\text{kojyā } b.\ \text{kojyā } c}{\text{jyā } b} + \frac{\text{jyā } c \text{ kojyā } A}{R};$$

which is equivalent to

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Again $\quad MH = ML + LH = ML + OV.$

Therefore by (1.1), (2.1) and (3)

$$\frac{\text{kojyā } a, \text{ kojyā } b}{\text{jyā } b} = \frac{\text{jyā } a.\ \text{jyā } (C - 90°)}{R} + \frac{R \text{ kojyā } c}{\text{jyā } b},$$

or R^2 *kojyā* $c = R$ *kojyā* a. *kojyā* b + *jyā* a *jyā* b *kojyā* C;

which is equivalent to

$$\cos c = \cos a \cos b + \sin a \sin b \cos C, \tag{i.1}$$

a formula similar to (*i*).

From the similar triangles OQQ' and OMM', we have

$$\frac{OQ}{OM} = \frac{QQ'}{MM'} = \frac{Q'O}{M'O}. \tag{4}$$

Therefore $OM.\ Q'O = OQ\ .\ OM' = OQ\,(MH - OV)$

$$= YN.\ HB - OV.\ R. \left[\because \frac{MH}{YN} = \frac{HB}{NO}\right]$$

Hence $\quad OM.\ Q'O = YN\,(RO' + O'V - RT) - OV.\ R;$

or $$\text{jyā } a.\ \text{jyā } (C - 90°) = \text{jyā } (90° - b) \left(\text{jyā } c + \frac{\text{kojyā } b.\ \text{kojyā } c}{\text{jyā } b} - \frac{\text{jyā } c.\ \text{utjyā } A}{R}\right) - \frac{R^2 \text{ kojyā } c}{\text{jyā } b};$$

or $-jy\bar{a}\ a.\ kojy\bar{a}\ C = \frac{kojy\bar{a}\ b.\ jy\bar{a}\ c}{R}(R - utjy\bar{a}\ A) - \frac{kojy\bar{a}\ c}{jy\bar{a}\ b}[R^2 - (kojy\bar{a}\ b)^2]$,

or $jy\bar{a}\ a.\ kojy\bar{a}\ C = kojy\bar{a}\ c.\ jy\bar{a}\ b - jy\bar{a}\ c.\ kojy\bar{a}\ b.\ kojy\bar{a}\ A$;

which is equivalent to

$$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A.$$

Since $MM' = BT$, we get from (4)

$$\frac{OM}{OQ} = \frac{BT}{QQ'}$$

Hence $$\frac{jy\bar{a}\ a}{R} = \frac{jy\bar{a}\ c\ \ jy\bar{a}\ A}{R\, jy\bar{a}\ C}$$

or $$\frac{jy\bar{a}\ a}{jy\bar{a}\ A} = \frac{jy\bar{a}\ c}{jy\bar{a}\ C}\ .$$

Similarly it can be proved that

$$\frac{jy\bar{a}\ c}{jy\bar{a}\ C} = \frac{jy\bar{a}\ b}{jy\bar{a}\ B}\ .$$

These are of course equivalent to

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}\ .$$

As an illustration of the application of the above formulae we take up the problem of the determination of the relation between the zenith distance (z), azimuth (ψ) and hour angle (H) of a heavenly body of known declination (δ) at a station whose terrestrial latitude is ϕ. In Fig. 15 let $NEQS$ denote the horizon, $NACS$ the meridian circle, KEK' the equator, RBR' the diurnal circle of the heavenly body (B), AFE the six o'clock circle, A the north pole and C the zenith. Then $a = z$, $b = 90° - \phi$, $c = 90° - \delta$, $\angle A = H$, $C = \psi$.

Substituting these values in the formulae (i), (i.1) and (ii), we obtain

$$\cos z = \sin \delta \sin \phi + \cos \delta \cos \phi \cos H$$

$$\sin \delta = \cos z \sin \phi + \sin z \cos \phi \cos \psi,$$

and

$$\sin z \cos \psi = \sin \delta \cos \phi - \cos \delta \sin \phi \cos H.$$

These equations were obtained by most of the Hindu astronomers.[1] It should however be made clear that the final results were arrived at by successive stages. The straight line DVD', the line of intersection of the diurnal circle with the horizon, is technically called *udayāsta-sūtra* ("the thread through the rising and setting points"), BM is called *śaṅku* ("gnomon"); BH *cheda* or *iṣṭahṛti* ("optional divisor"), MH *śaṅkutala*, BG (=the *jyā* in the diurnal circle of the complement of the hour angle) *kalā*, GH (=the *jyā* of the arc of the diurnal circle intercepted between the horizon and the six o'clock circle) *kujyā* or *kṣitijyā* ("earth-sine"), HL(=*jyā* ED) *agrā*, ML *bāhu*, OM *dṛgjyā* ("*jyā* of the zenith distance"), the angle EAD *cara* ("the ascensional difference"). The existence of these technical terms proves conclusively that the Hindus actually made the constructions contemplated above. They recognise the angle MBH to be equal to the latitude of the observer's station.

The *Sūrya-siddhānta* says:

"The *R*sine of the declination multiplied by the *palabhā* (=12 tan ϕ) and divided by 12 gives the *kujyā* ("the earth-sine"); that multiplied by the radius and divided by the radius of the diurnal circle will give the *jyā* whose arc will be the *cara* ("ascensional difference")."[2]

$$\text{kujyā} = \frac{\text{jyā}\,\delta \times 12\,\text{jyā}\,\phi}{12 \times \text{kojyā}\,\phi}$$

$$\text{carajyā} = \frac{\text{kujyā} \times R}{\text{kojyā}\,\delta}$$

Again,

"The radius plus the *carajyā* in the northern hemisphere, or minus it in the southern hemisphere is called *antyā*; subtract from it the versed *R*sine of the hour angle; (the remainder) multiplied by the radius of the diurnal circle and divided by the radius will be the *cheda*; that multiplied by the *R*sine of the co-latitude and divided by the radius will be the *śaṅku*; subtract the square of it from the square of the radius; the square-root of the remainder will be the *jyā* of the zenith distance."[3]

$$R \pm \text{carajyā} = \text{antyā},$$

$$\frac{(\text{antyā} - \text{utjyā}\,H) \times \text{kojyā}\,\delta}{R} = \text{cheda},$$

[1] *PSi*, iv. 42-4; *SūSi*, iii. 28-31, 34-5; *BrSpSi*, iii. 25-40, 54-6; *SiŚi*, *Graha*, iii. 50-52; etc.
[2] *SūSi*, ii. 61; also *Ā*, iv. 26; *ŚiDVṛ*, ii. 17; *BrSpSi*, ii. 57-60; *PSi*, iv. 26-7; *SiŚi*, *Graha*, ii. 48.
[3] *SūSi*, iii. 34-6.

$$\frac{cheda \times kojy\bar{a}\ \phi}{R} = \acute{s}a\dot{n}ku,$$

and $$\sqrt{R^2 - (\acute{s}a\dot{n}ku)^2} = jy\bar{a}\ z,$$

Therefore, in the northern hemisphere,

$$kojy\bar{a}\ z = \acute{s}a\dot{n}ku$$

$$= \frac{kojy\bar{a}\ \delta\ kojy\bar{a}\ \phi}{R^2}\left(R + R\frac{jy\bar{a}\ \delta\ jy\bar{a}\ \phi}{kojy\bar{a}\ \delta\ kojy\bar{a}\ \phi} - utjy\bar{a}\ H\right)$$

or $$R^2\ kojy\bar{a}\ z = R\ jy\bar{a}\ \delta \cdot jy\bar{a}\ \phi + kojy\bar{a}\ \delta \cdot kojy\bar{a}\ \phi \cdot kojy\bar{a}\ H;$$

which is of course equivalent to

$$\cos z = \sin\delta \sin\phi + \cos\delta \cos\phi \cos H.$$

Again it has been said that[1]

$$\acute{s}a\dot{n}kutala \mp b\bar{a}hu = agr\bar{a},$$

the negative or positive sign being taken according as the heavenly body is in the northern or southern hemisphere. Further[2]

$$agr\bar{a} = \frac{R\ jy\bar{a}\ \delta}{kojy\bar{a}\ \phi} \text{ and } b\bar{a}hu = -\frac{jy\bar{a}\ z\ kojy\bar{a}\ \psi}{R};$$

Also[3]

$$\acute{s}a\dot{n}kutala = \frac{\acute{s}a\dot{n}ku \times jy\bar{a}\ \phi}{kojy\bar{a}\ \phi}$$

Hence substituting the values

$$-\frac{jy\bar{a}\ z\ kojy\bar{a}\ \psi}{R} = \frac{\acute{s}a\dot{n}ku \times jy\bar{a}\ \phi}{kojy\bar{a}\ \phi} - \frac{R\ jy\bar{a}\ \delta}{kojy\bar{a}\ \phi}$$

$$= \frac{jy\bar{a}\ \phi\ kojy\bar{a}\ \delta}{R^2}\left\{R + R\frac{jy\bar{a}\ \phi \cdot jy\bar{a}\ \delta}{kojy\bar{a}\ \phi\ kojy\bar{a}\ \delta} - utjy\bar{a}\ H\right\} - \frac{R\ jy\bar{a}\ \delta}{kojy\bar{a}\ \phi}$$

$$= \frac{jy\bar{a}\ \phi\ kojy\bar{a}\ \delta\ kojy\bar{a}\ H}{R^2} - \frac{jy\bar{a}\ \delta \cdot kojy\bar{a}\ \phi}{R},$$

[1] *Ibid*, iii. 23–4
[2] *SūSi*, iii. 27; *PSi*, iv. 39; *Ā*, iv. 30; *BrSpSi*, xxi. 61.
[3] *Ā*, iv. 28, 29; *BrSpSi*, iii. 65, xxi. 63.

which is of course equivalent to

$$\sin z \cos \psi = \sin \delta \cos \phi - \cos \delta \sin \phi \cos H.$$

Expansion of Trigonometrical Functions

Remarkable work on the expansion of trigonometrical functions, sin θ, cos θ, $\tan^{-1}\theta$, etc., was done in India by the astronomers of Kerala in the fourteenth, fifteenth and sixteenth centuries A.D. It will be discussed in another article which will be devoted to the "Calculus".

Abbreviations

Ā	*Āryabhaṭīya*
ĀpŚlSū	*Āpastamba-Śulba-sūtra*
BCMS	*Bulletin of the Calcutta Mathemetical Society*
BrSpSi	*Brāhma-sphuṭa-siddhānta*
DhGr	*Dhyānagrahopadeśa*
GrL	*Graha-lāghava*
GSS	*Gaṇita-sāra-saṅgraha*
IJHS	*Indian Journal of History of Science*
JIMS	*Journal of the Indian Mathematical Society*
KK	*Khaṇḍa-khādyaka* (Bina Chatterjee's edition)
KKau	*Karaṇa-kaustubha*
KPr	*Karaṇa-prakāśa*
KR	*Karaṇa-ratna*
L	*Līlāvatī* (Ānandāśrama edition)
LBh	*Laghu-Bhāskarīya*
LMā	*Laghu-mānasa*
MBh	*Mahā-Bhāskarīya*
MSi	*Mahā-siddhānta*
PSi	*Pañca-siddhāntikā*
ŚiDVṛ	*Śiṣya-dhī-vṛddhida*
SiSā	*Siddhānta-sārvabhauma*
SiŚe	*Siddhānta-śekhara*
SiŚi	*Siddhānta-śiromaṇi*
SiTVi	*Siddhānta-tattva-viveka*
SK	*Sumati-karaṇa*
SMT	*Sumati-mahā-tantra*
SūSi	*Sūrya-siddhānta*
VSi	*Vaṭeśvara-siddhānta*
VVSi	*Vṛddha-Vaśiṣṭha-siddhānta*
	Yantra-rāja

Indian Journal of History of Science, **19** (2): 95—104 (1984)

USE OF CALCULUS IN HINDU MATHEMATICS

BIBHUTIBHUSAN DATTA AND AWADESH NARAYAN SINGH

Revised by

KRIPA SHANKAR SHUKLA

Argara, Hussaingunj Crossing
Lucknow 226 001

1. DIFFERENTIAL CALCULUS

A Controversy

Attention was first drawn to the occurrence of the differential formula

$$\delta (\sin \theta) = \cos \theta \, \delta \theta$$

in Bhāskara II's (1150) *Siddhānta Śiromaṇi* by Pandit Bapu Deva Sastri[1] in 1858. The Pandit published a summarised translation of the passages which involve the use of the above formula. His summary was defective in so far as it did not bring into prominence the idea of the infinitesimal increment which underlies Bhāskara's analysis. Without making clear to his readers, the full significance of Bhāskara's result, the Pandit made the mistake of asserting—what was plain to him—that Bhāskara was fully acquainted with the principles of the differential calculus.

The Pandit was adversely criticised by Spotiswoode[2], who without consulting the original on which the Pandit based his conclusions, remarked (1) that Bapu Deva Sastri had overstated his case in saying that Bhāskarācārya was fully acquainted with the principles of the differential calculus, (2) that there was no allusion to the most essential feature of the differential calculus, viz. the infinitesimal magnitudes of the intervals of time and space therein employed, and (3) that the approximative character of the result was not realized.

Since the above controversy took place no serious investigation of the subject seems to have been made by any scholar[3]. In order that the reader may be better able to judge the merit of the Hindu claim to the invention of the differential calculus, it is desirable that the problems which required the use of the above differential formula be stated first.

Problems in Astronomy

The calculation of eclipses is one of the most important problems of Astronomy. In ancient days this problem was probably more important than it is now, because the

exact time and duration of the eclipses could not be foretold on account of lack of the necessary mathematical equipment on the part of the astronomer. In India, the Hindus observed fast and performed various other religious rites on the occasion of eclipses. Thus their calculation was a matter of national importance. It afforded the Hindu astronomer a means of demonstrating the accuracy of his science and his own ability to the public who patronised him. The problem of the calculation of conjunction of planets and occultation of stars was equally important both from scientific as well as religious view points.

In problems of the above nature it is essential to determine the true instantaneous motion of a planet or star at any particular instant. This instantaneous motion was called by the Hindu astronomers *tāt-kālika-gati*. The formula giving the *tāt-kālika-gati* (instantaneous motion) is given by Āryabhaṭa and Brahmagupta in the following form:

$$u'-u=v'-v\pm e\ (\sin w'-\sin w) \qquad (i)$$

where u, v, w are the true longitude, mean longitude, mean anomaly respectively at any particular time and u', v', w' the values of the respective quantities at a subsequent instant; and e is the eccentricity or the sine of the greatest equation of the orbit. The *tāt-kālika-gati* is the difference $u'-u$ between the true longitudes at the two positions under consideration. Āryabhaṭa and Brahmagupta used the sine table to find the value of ($\sin w'-\sin w$). The sine table used by them was tabulated at intervals of $3°45'$ and thus was entirely unsuited for the purpose. To get the values of sines of angles, not occurring in the table, recourse was taken to interpolation formulae, which were incorrect because the law of variation of the difference was not known.

A Differential Formula

Mañjula (932) was the first Hindu astronomer to state that the difference of the sines,

$$\sin w'-\sin w=(w'-w)\cos w,$$

where $(w'-w)$ is small.

He says:

"True motion in minutes[4] is equal to the cosine (of the mean anomaly) multiplied by the difference (of the mean anomalies) and divided by the *cheda*[5], added or subtracted contrarily (to the mean motion)."[6]

Thus according to Mañjula formula (*i*) becomes

$$u'-u=v'-v\pm e\ (w'-w)\cos w, \qquad (ii)$$

which, in the language of the differential calculus, may be written as

$$\delta u=\delta v\pm e\cos\theta\,\delta\theta.$$

We cannot say exactly what was the method employed by Mañjula to obtain formula (*ii*). The formula occurs also in the works of Āryabhaṭa II (950)[7]. Bhāskara II (1150),[8] and later writers. Bhāskara II indicates the method of obtaining the differential of sine θ. His method is probably the same as that employed by his predecessors.

Proof of the Differential Formula

Let a point P (*See* Fig. 1) move on a circle. Let its position at two successive intervals be denoted by P and Q. Now, if P and Q are taken very near each other, the direction of motion in the interval PQ is the same as that of the tangent at P. Let PT be measured along the tangent at P equal to the arc PQ. Then PT would be the motion of the point P if its velocity at P had not changed direction.

Fig 1

Discussing the motion of planets, Bhāskarācārya says: "The difference between the longitudes of a planet found at any time on a certain day and at the same time on the following day is called its (*sphuṭa*) *gati* (true rate of motion) for that interval of time."

"This is indeed rough motion (*sthūlagati*). I now describe the fine (*sūkṣma*) instantaneous (*tāt-kālika*) motion[9]. The *tātkālika-gati* (instantaneous motion) of a planet is the motion which it would have, had its velocity during any given interval of time remained uniform."

During the course of the above statement, Bhāskara II observes that the *tāt-kālika-gati* is *sūkṣma* ("fine" as opposed to rough), and for that the interval must be taken to be very small, so that the motion would be very small. This small interval of time has been said to be equivalent to a *kṣaṇa*[10] which according to the Hindus is an infinitesimal interval of time (immeasurably small).[11] It will be apparent from the above that Bhāskara did really employ the notion of the infinitesimal in his definition of *Tāt-kālika-gati*.

But in actual practice, the intervals that are considered are not infinitesimal. How are we, then, to apply the notion of *tāt-kālika-gati* to actual problems? The answer to the above question is given by Bhāskara II as follows:

In equation (*i*) we have to consider the sine-difference ($\sin w' - \sin w$). Let an arc of 90° be divided into n parts each equal to A, and let us consider the sine differences $R(\sin A - \sin O)$, $R(\sin 2A - \sin A)$, $R(\sin 3A - \sin 2A)$........These differences are termed *Bhogya khaṇḍa*. Bhāskara II says: "These are not equal to each other but gradually decrease, and consequently while the increase of the arc is uniform, the increment of the sine varies—on account of deflection of the arc."

In the figure given above, let the arc $PQ = A$. Then

$$R\ (\sin\angle BOQ - \sin\angle BOP) = QN - PM = Qn$$

which is the *Bhogya Khaṇḍa*. Bhāskara introduces the notion of *Tāt-kālika Bhogya Khaṇḍa* (instantaneous sine difference) in order to find the variation of the sine at P. According to him if the arc BP instead of being deflected towards Q, be increased in the direction of the tangent, so that $PT = PQ = A$, then $TS - PM = Tr$ is the *Tāt-kālika Bhogya Khaṇḍa* of the sine PT, i.e. the "instantaneous sine difference". By having recourse to this artifice Bhāskara II avoids the use of the infinitesimal in his analysis. It should be borne in mind that the "instantaneous sine difference" for a finite arc PQ, is a purely artificial quantity created with a special end in view, and is different from the actual "sine difference" $R(\sin BOQ - \sin BOP)$.

Now from the similar triangles PTr and PMO, we at once derive the proportion[12]

$$R : PT :: R\cos w : Tr \ldots\ldots\ldots\ldots \qquad (iii)$$

$$\therefore\quad Tr = PT\cos w.$$

But

$$Tr = R(\sin w' - \sin w) \text{ and } PT = R\,(w' - w)$$

$$(\sin w' - \sin w) = (w' - w)\cos w.$$

Thus the *Tāt-kālika Bhogya Khaṇḍa* (the instantaneous sine difference) in modern notation is

$$\delta\ (\sin\theta) = \cos\theta\,\delta\,\theta.$$

This formula has been used by Bhāskara to calculate the *ayana-valana* ("angle of position").[13]

If the above were the only result occurring in Bhāskara II's work, one would be justified in not accepting the conclusions of Pandit Bapu Deva Sastri. There is however other evidence in Bhāskara II's work to show that he did actually know the principles of the differential calculus. This evidence consists partly in the occurrence of the two most important results of the differential calculus:

(*i*) He has shown that when a variable attains the maximum value its differential vanishes.

(*ii*) He shows that when a planet is either in apogee or in perigee the equation of the centre vanishes, hence he concludes that for some intermediate posi-

tion the increment of the equation of centre (*i.e.* the differential) alos vanishes.[14]

The second of the above results is the celebrated Rolle's Theorem, the mean value theorem of the differential calculus.

Remarks

The use of a formula involving differentials in the works of ancient Hindu mathematicians has been established beyond the possibility of any doubt. That the motions of instantaneous variation and that of motion entered into the Hindu idea of differentials as found in works of Mañjula, Āryabhaṭa II and Bhāskara II is apparent from the epithet *Tāt-kālika* (instantaneous) *gati* (motion) to denote these differentials. The main contribution of Bhāskara II to the theory of these differentials, which were already worked out by his predecessors, seems to be his proof of the formula by the rule of proportion without actually using the infinitesimal or varying quantities. He has, however, made it quite clear that the differentials give true results only when very small variations are concerned.

Nīlakaṇṭha's Result

Nīlakaṇṭha (*c.* 1500) in his commentary on the *Āryabhaṭīya* has given proofs, on the theory of proportion (similar triangles) of the following results.

(1) The sine-difference $\sin(\theta+\delta\theta)-\sin\theta$ varies as the cosine and decreases as θ increases.
(2) The cosine-difference $\cos(\theta+\delta\theta)-\cos\theta$ varies as the sine negatively and numerically increases as θ increases.

He has obtained the following formulae:

$$(a)\quad \sin(\theta+\delta\theta)-\sin\theta=2\sin\frac{\delta\theta}{2}\cos\left(\theta+\frac{\delta\theta}{2}\right)$$

$$(b)\quad \cos(\theta+\delta\theta)-\cos\theta=-2\sin\frac{\delta\theta}{2}\sin\left(\theta+\frac{\delta\theta}{2}\right).$$

The above results are true for all values of $\delta\theta$ whether big or small. There is nothing new in the above results. They are simply expressions as products of sine and cosine differences.

But what is important in Nīlakaṇṭha's work is his study of the second differences. These are studied geometrically by the help of the property of the circle and of similar triangles. Denoting by $\triangle_2(\sin\theta)$ and $\triangle_2(\cos\theta)$, the second differences of these functions, Nīlakaṇṭha's results may be stated as follows:

(1) The difference of the sine-difference varys as the sine negatively and increases (numerically) with the angle.

(2) The difference of the cosine-difference varys as the cosine negatively and decreases (numerically) with the angle.

For $\triangle_2$ (sin θ), Nīlakantha[15] has obtained the following formula

$$\triangle_2 \ (\sin \theta) = -\sin \theta . \left(2 \sin \frac{\triangle \theta}{2}\right)^2$$

Besides the above, Nīlakaṇṭha, has made use of a result involving the differential of an inverse sine function.[16] This result, expressed in modern notation, is

$$\delta \ (\sin^{-1} e \sin w) = \frac{e \cos w}{\sqrt{1 - e^2 \sin^2 w}} \ \delta w$$

In the writings of Acyuta (1550-1621 A.D.) we find use of the differential of a quotient[17] also

$$\delta \left[\frac{e \sin w}{1 \pm e \cos w} \right] = \frac{e \cos w \pm (e \sin w)^2/(1 \pm e \cos w)}{1 \pm e \cos w} \ \delta w$$

2. Method of Infinitesimal-Integration

Surface of the Sphere

For calculating the area of the surface of a sphere Bhāskara II (1150) describes two methods which are almost the same as we usually employ now for the same purpose.

First Method: "Make a spherical ball of clay or of wood. On it take a (vertical) circumference circle and divide this into 21600 parts. Mark a point on the top of it. With that point as the centre and with the radius equal to the 96th part of the circumference, i.e. to 225′, describe a circle. Again with same point as the centre with twice that arc as radius describe another circle; with thrice that another circle; and so on upto 24 times. Thus there will be 24 circles in all. The radii of these circles will be the *jyā* 225′ (=*R*sin 225′), etc. From them the lengths of the circles can be determined by proportion. Now the length of the extreme circle is 21600′ and its radius is 3438′. Multiplying the *R*sines (of 225′, 450′, etc.) by 21600 and dividing by 3438, we shall obtain the lengths of the circles. Between two and two of these circles there lies annular strips and there are altogether 24 such. They will be many more in case of many *R*-sines being taken into consideration (*bahujyā-pakṣe-vahūni syuḥ*). In each annulus considering the larger circle at the lower end as the base and the smaller circle at the top as the face and 225′ as the altitude (of the trapezium), find its area by means of the rule 'half the sum of the base and the face multiplied by the altitude etc'.[18] Similarly the areas of all the annular figures severally can be found. The sum of all these areas is equal to the area of the surface of half of the sphere. So twice that is the area of the

surface of the whole sphere. And that is equal to the product of the diameter and the circumference."[19]

In other words, if T_n denotes the nth *jyā* (or *R*sine), C_n the circumference of the corresponding circle, A_n the area of the nth annulus and S the area of the surface of the sphere, then we shall have

$$C_n = \frac{21600}{3438} T_n$$

$$A_n = \frac{C_n + C_{n+1}}{2} \times 225$$

$$= \frac{225 \times 21600}{2 \times 3438} (T_n + T_{n+1}).$$

Therefore,

$$\tfrac{1}{2} S = \Sigma A_n$$

$$= \frac{225 \times 21600}{2 \times 3438} \sum (T_n + T_{n+1})$$

the summation being taken so as to include all the *R*sines in a quadrant of the circle. Since there are ordinarily 24 *R*sines in a Hindu trigonometrical table, we have

$$\tfrac{1}{2} S = \frac{225 \times 21600}{3438} \sum (T_1 + T_2 + \ldots\ldots + T_{23} + \tfrac{1}{2} T_{24})$$

$$= \frac{21600 \times 225 \times 52513}{3438}$$

$$= 21600 \times 3436.7 \ldots\ldots\ldots\ldots$$

Hence approximately

$$S = 21600 \times 2 \times 3437$$

Bhāskara II states:

Area of the surface = circumference × diameter.

Second Method: "Suppose the (horizontal) circumference-circle on the surface of the sphere to be divided into parts as many as four times the number of *R*sines (in a quadrant). As the surface of an emblic myrobalan is seen divided into *vapras* (i.e. lunes) by lines passing through its face (or top) and bottom, so the surface of the sphere should be divided into lunes by vertical circles as many as the parts of the above mentioned (horizontal) circumference-circle. Then the area of each lune should be determined by (breaking it up into) parts. And this area of a lune is equal to the sum of all the *R*sines diminished by half the radius and divided by the semi-radius. Since that is

again equal to the diameter of the sphere, so it has been said that the area of the surface of a sphere is equal to the product of its circumference and diameter."[20]

The method has been further elucidated by him in his gloss thus:

"As many as are *R*sines in the table of any particular work selected, take four times that number, and suppose the (horizontal) circumference-circle on the sphere is to be divided into, as many parts. Like the natural lines seen on the surface of a round emblic myrobalan passing through its face and base and thus dividing it into lunes, draw circles on the surface of the given sphere, passing through its top and bottom and thereby dividing it into lunes as many as the number of parts into which the (horizontal) circumference-circle is divided. Next the area of each lune has to be determined. It can be done thus: For instance in the *Dhīvṛddhida*,[21] there are 24 *R*sines. So suppose the (horizontal) circumference-circle measures 96 cubits. On drawing the vertical circles through every cubit, there will be as many lunes. Then the upper half of any one lune on drawing the transverse arcs at distances of every cubit, will be divided into portions equal to the number of *R*sines, that is, 24. The lengths of these transverse lines will be obtained by dividing the *R*sines severally by the radius. Of these the lowest line measures one cubit; but the upper and upper ones are a little smaller and smaller according to the *R*sines. But the altitude is all along one cubit in length. Now by finding the area of each portion in accordance with the rule, "half the sum of the top and the base multiplied by the altitude etc: they should be added together. This sum gives the area of half a lune; twice that is the area of a lune. For the determination of that the rule is, "the sum of all the *R*sines minus half the radius etc." Now the sum of all the *R*sines, 225 etc., is 54233.[22] This diminished by the semi-radius becomes 52514. Dividing the result by the semi-radius we get the area of each lune as 30;33. Now 30;33 is equal to the diameter of a circle whose circumference measures 96. And as the number of lunes is equal to the number of portions of the circumference it is consequently proved that the area of the surface of a sphere is equal to the product of its circumference and diameter".

If l_n denotes the length of the nth transverse arc, we have

$$l_n = \frac{T_n \times 1}{R}.$$

Therefore,

$$\text{area of a lune} = 2 \times \sum \frac{1}{2}(l_n + l_{n+1}) \times 1$$

$$= 2 \sum \frac{1}{2R}(T_n + T_{n+1})$$

the summation being taken so as to include all the *R*sines. Hence

$$\text{area of a lune} = 2 \times \frac{1}{R}\left(T_1 + T_2 + \ldots\ldots + T_{23} + \frac{1}{2}T_{24}\right)$$

$$= \frac{1}{R/2}\left((T_1 + T_2 + \ \ldots\ldots\ldots + T_{24} - \frac{R}{2} \right)$$

$$= 30;32,94 \ldots\ldots\ldots$$

Now $96 \times \frac{1250}{3927} = 30;33,46 \ldots\ldots\ldots$

Hence the area of a lune is numerically equal to the diameter of the sphere. As the number of lunes is equal to the number of parts of the circumference of the sphere, we get

Area of the surface $=$circumference$\times$diameter.

Volume of the Sphere

To find the volume of a sphere Bhāskara II states the following method:

"Consider on the surface of the sphere pyramidal excavations, each of a base of an unit area having unit sides and of a depth equal to the radius, as many as the number of units of area in the surface. The apices of these pyramids meet at the centre of the sphere. The sum of the volumes of the pyramids is equal to the volume of the sphere. So it is proved (that the volume of a sphere is equal to the sixth part of the product of the surface area and diameter)[23].

The above results are the nearest approach to the method of the integral calculus in Hindu Mathematics. It will be observed that the modern idea of the "limit of a sum" is not present. This idea, however, is of comparatively recent origin so that credit must be given to Bhāskara II for having used the same method as that of the integral calculus, although in a crude form.

REFERENCES

[1]*JASB* (=*Journal of the Asiatic Society of Bengal*), Vol. 27, 1858, pp. 213-6.

[2]*JARS*, Vol. 17, 1860, pp. 21-2.

[3]Except for a paper by P. C. Sen Gupta in the *Journal of the Department of Letters, Calcutta University*, Vol. XXII (1931). Recently A. K. Bag has included this topic in his book "Mathematics in Ancient and Medieval India", Chaukhambha Orientalia, Varanasi, 1979.

[4]This clearly shows that the formula is intended for use when difference is small, the result being expressible in minutes.

[5]Here *cheda* (divisor)$=1/e$. According to Hindu astronomers $1/e=360/P$, where P is the periphery of the epicycle.

[6]*Laghu-mānasa*, ii. 7.

[7]*MSi*=*Mahā-Siddhānta*), iii. 15f.

[8]*SiŚi* (=*Siddhānta-Śiromaṇi*), *Gaṇitādhyāya, Spaṣṭādhikāra*, 36-7.

[9]*SiŚi* (=*Siddhānta-Śiromaṇi*), *Gaṇitādhyāya, Spaṣṭādhikāra*, 36 (c-d).

[10]The smallest unit of time, according to Bhāskara II is a *truti* (*SiŚi, Gaṇita, Madhyamādhikāra, Kālamānādhyāya*, 6), which is equivalent to 1/33750 of a second. The *Kṣaṇa* is smaller, in fact the smallest interval of time that can be imagined.

[11]These remarks are made with reference to the motion of the moon. As the motion of the moon is comparatively quicker, so the *tāt-kālika-gati* will not give correct result unless the time interval is taken small enough.

[12]It should be noted that for the purpose of the following proof, it is immaterial, whether we take PQ small or not, because it is PT that we are considering and not PQ. Bhāskara actually takes the $\angle POQ=(3\frac{3}{4})^\circ=225'$ for exhibiting equation (iii). The notion of the infinitesimal is here involved in the definition of *Tāt-kālika Bhogya Khaṇḍa*.

[13]*SiŚi, Golādhyāya, Grahaṇa, Grahaṇa-vāsanā*; see also Sen Gupta, l. c., p. 11 ff.

[14]These results occur in the *Golādhyāya, Spaṣṭādhikāra vāsanā* of the *Siddhānta Śiromaṇi*, and were first noted by Sudhakara Dvivedi.

[15]This together with the results given above are proved by Nīlakaṇṭha in the commentary on the *Āryabhaṭīya*, ii. 12.

[16]*Cf. Tantrasaṃgraha*, ii. 53-4.

[17]*Cf. Sphuṭa-nirṇaya-tantra*, iii, 19-20; *Karaṇottama*, ii. 7.

[18]The rule quoted here for finding the area of a trapezium is that given by Śrīdhara (*Triś*, R. 42) Bhāskara II's rule is defined slightly differently (vide *L*, p. 44).

[19]*SiŚi, Gola, Bhuvanakośa*, verses 55-7 (Gloss.)

[20]*Ibid*. verses 58-61.

[21]That is *Śiṣyadhīvṛddhida* of Lalla

[22]According to Lalla the sum of the *R* sines is 54233.

[23]*SiŚi, Gola, Bhuvanakośa*, Vss. 58-61, (Gloss).

Indian Journal of History of Science, **27**(1), 1992

MAGIC SQUARES IN INDIA

BIBHUTIBHUSAN DATTA AND AWADHESH NARAYAN SINGH
(Revised by KRIPA SHANKAR SHUKLA)

Hussainganj Crossing, Behind Lakshman Bhawan, Lucknow

(Received 22 August 1991)

A square containing an equal number of cells in each row and each column is called a magic square, when the total of numbers in the cells of each row, each column and each diagonal happens to be the same. Magic squares have been known in India from very early times. It it believed that the subject of magic squares was first taught by Lord Śiva to the magician Maṇibhadra. Magic squares are said to have magical properties and were used in various ways by the Hindus as well as the Jainas.But the mathematics involved in the construction of magic squares and other magic figures was first systematically and elaborately discussed by the mathematician Nārāyaṇa (AD 1356) in his *Gaṇitakaumudī*. Some of his methods were unknown in the west and were recently discovered by the efforts of several scholars. The present article, besides giving a brief history of magic squares, explains the methods given by Nārāyaṇa and other Hindu writers for the construction of magic squares of various types

ORIGIN AND EARLY HISTORY

Little is known as regards the origin of magic squares and other figures. Hindu tradition assigns them to God Śiva. Nārāyaṇa (1356) says that the subject of progression, of which magic squares form a part, was taught by Śiva to Maṇibhadra*, the magician. The earliest unequivocal occurrence of magic squares is found in a work called *Kakṣapuṭa* composed by the celebrated alchemist and philosopher Nāgārjuna who flourished about the 1st century AD. One of the squares in this work is called *Nāgārjunīya* after him; so there can be no doubt that he did really construct some squares. The squares given by Nāgārjuna are all 4 × 4 squares, and some of these seem to have been known before him. The easier case of 3 × 3 square must have also been known earlier to Nāgārjuna. Another square is found in a work of Varāhamihira (d. 587 AD).

4 × 4 magic squares are considered to possess magical properties and are supposed to bring luck when worn as amulets. They are found on gates of buildings, on the walls where shopkeepers transact their business and on the covers of calendars used by astrologers even to this day. A 4 × 4 square occurs in a Jaina inscription of the 11th century, found in the ancient town of Khajuraho.

---

*Reference to Maṇibhadra Yakṣa occurs in the Buddhist work *Saṃyukta-nikāya* (i.10,4) and the Jaina work *Sūrya-prajñapti*. See D.N. Shukla, *Pratimā-Vijñāna* (in Hindi), p. 51.

A systematic study of magic figures was taken up by Nārāyaṇa, who in his *Gaṇita Kaumudī* (1356) gives general methods for the construction of all sorts of magic squares with the principles governing such constructions. He seems to have been the first to conceive of other figures in which numbers may be arranged so as to possess properties similar to those of magic squares. An account of the methods of constructing magic squares given by the authors mentioned above and also by other Hindu writers is given in this article.

It is the opinion of some historians of mathematics that magic squares first originated in China. This opinion is based on the occurrence of a square, filled with white and black dots, in the introduction of a Chinese work, the *I-king*. The square is called the *Loh Shu*, and is said to have come down to us from the time of the great emperor Yu (*c.* 2200 BC). According to Chinese tradition, the white dots denote odd numbers and the black dots even ones, and it has been conjectured that the *Loh Shu* represents the square

4	9	2
3	5	7
8	1	6

Fig. 1

But to consider the *Loh Shu* as a magic square is to force upon it an interpretation which it originally did not possess. Arrangements of white and black dots in the figure of a square are met with elsewhere in the literature of the Chinese. One such arrangement represents the river *Ho* and has nothing to do with magic squares[1].

The first unequivocal appearance of the *Loh Shu* in the form of a magic square is found in the writings of Tsai Yuan-Ting[2] who lived from 1135 to 1198 AD. Magic squares occur also in the writings of Hebrew[3] and Arab[4] scholars about the same period, while in India they were used much earlier. It would thus appear that the Chinese claim to the invention of magic squares is not well founded[5].

NĀGĀRJUNA SQUARES

In his *Kakṣapuṭa*, Nāgārjuna (100) gives rules for the construction of 4×4 squares with even as well as odd totals[6]. These rules consist partly of mnemonic verses in which numbers are expressed in alphabetic notations. The general direction is

Arka	*Indunidhā*	*Nārī*	*tena*	*lagna*	*viuāsanam*
0 1	0 8 0 9	0 2	6 0	3 0	4 0 7 0

By inserting these values in the successive cells (of the 4×4 square) leaving blanks for zero, we get the primary skeleton

	1		8
	9		2
6		3	
4		7	

Fig. 2

The eight blank cells can be filled up in such a way as to give even as well as odd totals. But the methods of filling up differ slightly in the two cases.

(i) *Even Total:* In order to have an even total, fill up, says Nāgārjuna, the blank cells by writing the difference between half of that total and the number in the alternate cell in a diagonal direction from the cell to be filled up. This direction may be upwards or downwards, right or left.

Taking the total to be 2n, where n is any integer, we thus get the complete magic square with even totals.

$n-3$	1	$n-6$	8
$n-7$	9	$n-4$	2
6	$n-8$	3	$n-1$
4	$n-2$	7	$n-9$

Total = 2n

Fig. 3

In this magic square, the totals of all the rows, horizontal, vertical, and diagonal, of every group of four forming a sub-square, and separated by such a sub-square, and of the corner four of the square and about a small square, are

equal. Another noteworthy feature of it is that each of its four minor squares has relation to others, as may be seen in Fig. 4.

$n-2$ $2n-10$ 10 $n+2$	$n+2$ $2n-10$ 10 $n-2$
$n-2$ 10 $2n-10$ $n+2$	$n+2$ 10 $2n-10$ $n-2$

Fig. 4

The above square is "continuous" according to the definition of Paul Carus; that is, "It may vertically as well as horizontally be turned upon itself and the rule holds good that wherever we may start four consecutive numbers in whatever direction, backward or forward, upward or downward, in horizontal, vertical or slanting lines, always yield the same sum....and so does any small square of 2×2 cells"[7]. Since the square cannnot be bent upon itself at once in two directions, the result is shown in Fig. 5, by extending the square in each direction by half its own size.

3	$n-1$	6	$n-8$	3	$n-1$	6	$n-8$
7	$n-9$	4	$n-2$	7	$n-9$	4	$n-2$
$n-6$	8	$n-3$	1	$n-6$	8	$n-3$	1
$n-4$	2	$n-7$	9	$n-4$	2	$n-7$	9
3	$n-1$	6	$n-8$	3	$n-1$	6	$n-8$
7	$n-9$	4	$n-2$	7	$n-9$	4	$n-2$
$n-6$	8	$n-3$	1	$n-6$	8	$n-3$	1
$n-4$	2	$n-7$	9	$n-4$	2	$n-7$	9

(ii) *Odd Total:* For an odd total, say 2n + 1, we are to fill up the blank cells by writing the difference between n and the number in the alternate cell in a diagonal direction from the cell to be filled up, when the latter number happens to be 1, 2, 3 or 4; or the difference between n + 1 and the number in the alternate cell in a diagonal direction from the cell to be filled up, if the latter number be 6, 7, 8 or 9. This direction may be, as in the previous case, upwards or downwards, right or left. Proceeding in this way, we get the complete magic squares having an odd total.

n−3	1	n−5	8
n−6	9	n−4	2
6	n−7	3	n−1
4	n−2	7	n−8

Total = 2n + 1
Fig. 6

In this case , the totals of all rows, horizontal, vertical and diagonal, of every group of four forming a square (except the group of the fifth, sixth, ninth and tenth cells, and that of the seventh, eighth, eleventh and twelfth cells), of the corner four of the square, and of the four about the corners of a small square are equal. The relation between the four minor squares in this case is not as complete as in the previous case.

n−2 2n−9 10 n+3	n+3 2n−9 10 n−2
n−1 10 2n−9 n+2	n+2 10 2n−9 n−1

Fig. 7

It is not a perfectly continuous square. The odd totals cannot be less than 19 in any case, and not less than 37 if the same number is not to appear more than once in the square. (See Fig.6)

A particular case of 4×4 squares with even total, 100, has been specially noted by Nāgārjuna. Its form differs from that which results on putting n=50 in the above general case, and further it does not contain the numbers from 1 to 9, except 6.

30	16	18	36
10	44	22	24
32	14	20	34
28	26	40	6

Total = 100

Fig. 8

This magic square has been called the Nāgārjunīya[8]. This special epithet will lead one to presume that this particular square was constructed by Nāgārjuna, while others described by him were recapitulations of former accomplishments.

Varahamihira Square

Varāhamihira (d. 587 AD) gives a form of 4×4 magic squares, viz.[9]

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

Total = 18

Fig. 9

in which the total is 18. It is, however, a particular case of the following:

n−7	3	n−4	8
5	n−1	2	n−6
4	n−8	7	n−3
n−2	6	n−5	1

Total = 2n
Fig. 10

2	n−6	5	n−1
n−4	8	n−7	3
n−5	1	n−2	6
7	n−3	4	n−8

Total = 2n
Fig. 11

Varāhamihira has called his square *Sarvatobhadra* ("Magic in all respects") and what are implied by that name, i.e., the special features of the square, have been pointed out fully by his commentator, Bhaṭṭotpala (966). Indeed it has properties similar to those of squares with even totals described by Nāgārjuna. The method of filling up the blank cells in the primary skeleton is the same.

	3		8
5		2	
4		7	
	6		1

Fig. 12

2		5	
	8		3
	1		6
7		4	

Fig. 13

The blank cells can be filled up so as to yield also an odd total; write the difference between n + 1 and the number in the alternate cell in a diagonal direction from the cell to be filled up, if the latter number happens to be 1, 2, 3 or

4, or the difference between n and the number in the alternate cell in a diagonal direction from the cell to be filled up, if the latter number happens to be 5, 6, 7 or 8.

n−7	3	n−3	8
5	n	2	n−6
4	n−8	7	n−2
n−1	6	n−5	1

Total = 2n + 1
Fig. 14

2	n−6	5	n
n−3	8	n−7	3
n−5	1	n−1	6
7	n−2	4	n−8

Total = 2n + 1
Fig. 15

Jaina Squares

In a Jaina inscription found amongst the ruins of the ancient town of Khajuraho occurs a magic square of 4 x4 cells of which the total is 34 (Fig. 16). It possesses all the special features of the Nāgārjuna squares. It belongs to the eleventh century of the Christian era. In the *Tijapapahutta Stotra* of the Jainas, we find another 4×4 magic square having a total of 170[10] (Fig. 17).

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Total = 34
Fig. 16

25	80	15	50
20	45	30	75
70	35	60	5
55	10	65	40

Total = 170
Fig. 17

The date of this square is uncertain. It is certainly not later than the fourteenth century, when a commentary on the above *Stotra* (hymn) was written. It is

probably a very old one. Its total 170 is closely connected with an ancient Jaina mythology about the appearance of their prophets.

NĀRĀYAṆA'S RESULTS

As has been already pointed out, the only Hindu work, known to us, which gives a systematic mathematical treatment of the construction of magic squares and other figures is the *Gaṇita Kaumudī* of Nārāyaṇa. Chapter XIV of the work is devoted to this subject, and we propose to give here a summarized version of this chapter, preserving the order of treatment and giving explanatory notes wherever necessary.

SUMMARY

In order to bring into prominence the remarkable achievements of Nārāyaṇa in the theory of magic squares, it is thought desirable to state briefly some of his most important results before entering into details. These results are:

1. Magic squares are of three types: (a) those which have 4n cells in a row, (b) those which have $4n+2$ cells in a row, and (c) those which have an odd number of cells in a row.

2. Series in arithmetical progression are used for the construction of these squares.

3. Magic squares can be made of as many series or groups of numbers as there are cells in a column.

4. Each series or group is composed of as many numbers as there are groups.

5. The common difference must be the same for each group.

6. The initial terms of the groups are themselves in A.P.

7. The numbers is a group, although belonging to an arithmetical progression, may be disarranged in various ways for the filling of the square.

8. The method of the knight's move for the construction of a $4n \times 4n$ square.

9. The method of superposition for the construction of $4n \times 4n$ squares.

10. The method of equi-spacing for the construction of $(4n+2) \times (4n+2)$ squares.

11. The method of superposition for odd squares.

12. A special method for odd squares.

13. The construction of a magic rectangle (*vitāna* or canopy).

14. The construction of magic circles, triangles, hexagons and various other figures, such as the altar, the diamond, etc.

PRELIMINARY REMARKS

According to Nārāyaṇa, magic squares may be classified into three groups: (1) *samagarbha*, (2) *viṣamagarbha*, and (3) *viṣama*. These terms are defined as follows:

> "If on dividing the *bhadrāṅka*[11] ("number of cells in a line of the square") by four, the remainder is zero, the magic square is said to be *samagarbha*. If the remainder is two, it is called *viṣamagarbhā*; and if the remainder is one or three, it is simply *viṣama*[12].

After giving the above classification, Nārāyaṇa remarks:

> "In the construction of magic squares, the arithmetical progression is used[13]. In relation to that (magic square) which is required to be constructed, first find the *initial term* and the *common-difference* (of a series in arithmetical progression, corresponding to the given sum and the number of cells)[14]. The sum divided by the *bhadrāṅka* ("number of the square") gives the *phala* ("total"). The number of terms to be taken in the progression is the number of *gṛha* ("cells") in the square[15]. If the number of cells *(koṣṭha)* is a square number, its root is called the *carana* ("foot" or "row"). Such are the technical terms used by Nārāyaṇa in his *bhadragaṇita* ("calculations relating to magic figures")[16].

The method of finding out the initial term and the common-difference of an arithmetical progression, given the sum and the number of terms, follows the above preliminary remarks[17].

THE 4 × 4 MAGIC SQUARES[18]

Assuming that the required arithmetical progression has been found out, Nārāyaṇa gives the following rule for filling the cells of the 4 × 4 square with the numbers occurring in the progression:

> "In the manner of the chess-board, place the numbers forming the progression, (taking them) two and two, in two connected cells as well as in alternate cells, in the direct and inverse order. (Then) by right and left knight's move fill the cells (of the square) with the numbers (taking them as they have been placed above). This method has also been stated by previous teachers for the

construction of the *samagarbha* magic square of sixteen cells. The numbers, in the horizontal cells, in the vertical cells as well as in the diagonal cells, added separately, give rise to the same total"[19].

Example from Nārāyaṇa:

"Friend, tell me, how a 4×4 magic square be filled up with the numbers beginning with unity and successively increasing by one, so that the horizontal, vertical and diagonal cells shall have the same sum".

Here the sum of the natural numbers from one to sixteen is $(16 \times 17)/2 = 136$. Therefore, the required total is $136/4 = 34$.

The numbers when written, two and two, in connected cells as well as alternate cells give Figs. 18 and 19.

(a_0)

1	2	3	4
8	7	6	5
9	10	11	12
16	15	14	13

Fig. 18

(b_0)

1	3	2	4
8	6	7	5
9	11	10	12
16	14	15	13

Fig. 19

Placing two and two in the direct and inverse orders, we get Figs. 20 and 21.

(a)

1	2	4	3
8	7	5	6
10	9	11	12
15	16	14	13

Fig. 20

(b)

1	3	4	2
8	6	5	7
11	9	10	12
14	16	15	13

Fig. 21

[Note that the numbers in the 4 cells on the right of the first two rows are reversed and the same is done with the numbers in the 4 cells on the left of the last two rows. This would probably be an easier method of stating the method.]

Taking the arrangement (a), and filling by right and left knight's move, the four squares depicted in Figs. 22-25 are obtained.

(a_1)

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

Fig. 22

(a_2)

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

Fig. 23

(a_3)

1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

Fig. 24

(a_4)

1	14	4	15
12	7	9	6
13	2	16	3
8	11	5	10

Fig. 25

The arrangement (b) similary gives the four squares as per Figs. 26-29.

(b_1)

1	8	13	12
15	10	3	6
4	5	16	9
14	11	2	7

Fig. 26

(b_2)

1	15	4	14
8	10	5	11
13	3	16	2
12	6	9	7

Fig. 27

(b_3)

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

Fig. 28

(b_4)

1	15	4	14
12	6	9	7
13	3	16	2
8	10	5	11

Fig. 29

The filling in of the numbers begins by putting 1 in the first cell. After the first row of numbers is exhausted, we begin by putting the first number of the second row (i.e., 8) in a contiguous cell, as in (a_1) or (a_2). [In order to make up the desired total, 10 is made to correspond to 8 and 15 to 1, as in the illustration above.] If the square be considered to be wrapped round a cylinder, the fourth cell is also contiguous to the first cell; hence the squares (a_3) and (a_4). It may be noted that (a_2) may be obtained by turning (a_1) through a right angle, whilst (a_3) and (a_4) may be obtained by wrapping (a_1) and (a_2) round a cylinder.

VARIETIES OF 4×4 MAGIC SQUARES

Nārāyaṇa remarks:

"Here, other 4×4 squares may be produced from a 4×4 square by turning four cells to make the numbers inverse"[20].

"In the rest of the *caraṇa* ("row") following the first cell, (by turning) four numbers produced in two connected pairs of cells, there result twenty-four varieties. And the same numbers arise from others separately"[21].

Example from Nārāyaṇa:

How many 4×4 squares can be formed out of the series of natural numbers from one to sixteen, and what are their forms?[22]

The numbers are placed according to the previous rule as per Figs. 30 and 31.

(a)

1	2	4	3
8	7	5	6
10	9	11	12
15	16	14	13

Fig. 30

(b)

1	3	4	2
8	6	5	7
11	9	10	12
14	16	15	13

Fig. 31

By turning four cells, i.e., the two connected pairs in the middle of the first two rows, and doing the same for the last two rows, we have Figs. 32 and 33

(a')

1	5	7	3
8	4	2	6
10	14	16	12
15	11	9	13

Fig. 32

(b')

1	5	6	2
8	4	3	7
11	15	16	12
14	10	9	13

Fig. 33

Performing the same operation on the two connected pairs of cells at the end in (a) and (b), we have Figs. 34 and 35.

(a″)

1	2	6	5
8	7	3	4
10	9	13	14
15	16	12	11

Fig. 34

(b″)

1	3	7	5
8	6	2	4
11	9	13	15
14	16	12	10

Fig. 35

The numbers in the arrangements (a′), (b′), (a″) and (b″) are filled in the 4×4 square in the same way as those of (a) or (b). Thus, there will be altogether 24 squares with 1 in the first cell. As there are sixteen numbers, so there can be 384 varieties of 4×4 squares, formed out of the series of natural numbers one to sixteen.

The twenty-four varieties with 1 in the first cell have been shown by Nārāyaṇa as per Fig. 36.

Example: In a certain 4×4 square, the total *(phala)* is 40, find the initial term and the common-difference. Also find them when the total is 64.

The equations giving the initial term (a) and the common difference (d) are:

(i) $10 - \frac{15}{2} d = a$, when the total is 40;

and

(ii) $16 - \frac{15}{2} d = a$, when the total is 64.

These give: for case (i) $a = -5, \ldots; d = 2, \ldots$

and for case (ii) $a = 1, -14, \ldots; d = 2, 4, \ldots$

NĀRĀYAṆA'S SQUARES

[1]

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

[2]

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

[3]

1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

[4]

1	14	4	15
12	7	9	6
13	2	16	3
8	11	5	10

[5]

1	8	13	12
15	10	3	6
4	5	16	9
14	11	2	7

[6]

1	15	4	14
8	10	5	11
13	3	16	2
12	6	9	7

[7]

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

[8]

1	15	4	14
12	6	9	7
13	3	16	2
8	10	5	11

[9]

1	8	10	15
14	11	5	4
7	2	16	9
12	13	3	6

[10]

1	14	7	12
8	11	2	13
10	5	16	3
15	4	9	6

[11]

1	15	10	8
14	4	5	11
7	9	16	2
12	6	3	13

[12]

1	14	7	12
15	4	9	6
10	5	16	3
8	11	2	13

1	2	4	3
8	7	5	6
10	9	11	12
15	16	14	13

1	3	4	2
8	6	5	7
11	9	10	12
14	16	15	13

1	5	7	3
8	4	2	6
13	9	11	15
12	16	14	10

[13]

1	8	11	14
15	10	5	4
6	3	16	9
12	13	2	7

[17]

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

[21]

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

[14]

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

[18]

1	12	6	15
8	13	3	10
11	2	16	5
14	7	9	4

[22]

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

[15]

1	14	11	8
15	4	5	10
6	9	16	3
12	7	2	13

[19]

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

[23]

1	15	10	8
12	6	3	13
7	9	16	2
14	4	5	11

[16]

1	15	6	12
14	4	9	7
11	5	16	2
8	10	3	13

[20]

1	12	6	15
14	7	9	4
11	2	16	5
8	13	3	10

[24]

1	12	7	14
15	6	9	4
10	3	16	5
8	13	2	11

1	5	6	2
8	4	3	7
13	9	10	14
12	16	15	11

1	2	6	5
8	7	3	4
10	9	13	14
15	16	12	11

1	3	7	5
8	6	2	4
11	9	13	15
14	16	12	10

Fig. 36.

The squares constructed, according to the rule given above, with the above values of a and d are shown in Figs. 37-39.

−5	9	19	17
21	15	−3	7
1	3	25	11
23	13	−1	5

Total = 40

Fig. 37

1	15	25	23
27	21	3	13
7	9	31	17
29	19	5	11

Total = 64

Fig. 38

−14	14	34	30
38	26	−10	10
−2	2	46	18
42	22	−6	6

Total = 64

Fig. 39

USE OF IRREGULAR SERIES

Instead of employing 16 numbers in arithmetical progression to fill up a 4×4 square, four different arithmetic series, with different initial terms but the same common-difference consisting of four terms each may be used[23]. Nārāyaṇa gives the following examples to illustrate this:

Examples from Nārāyaṇa[24].

(a) To construct 4 × 4 magic squares with total 40.

In this case, the *caraṇas* (rows, i.e., the arithmetic progressions of which each has as many terms as there are cells in a row) may be supposed to be:

(i)	1	2	3	4	(ii)	1	2	3	4	(iii)	2	3	4	5
	6	7	8	9		5	6	7	8		6	7	8	9
	11	12	13	14		12	13	14	15		11	12	13	14
	16	17	18	19		16	17	18	19		15	16	17	18

Now filling up the cells by the same method as before, we get Figs. 40-42.

1	9	16	14
17	13	2	8
4	6	19	11
18	12	3	7

Total = 40

Fig. 40

1	8	16	15
17	14	2	7
4	5	19	12
18	13	3	6

Total = 40

Fig. 41

2	9	15	14
16	13	3	8
5	6	18	11
17	12	4	7

Total = 40

Fig. 42

(b) To construct 4 × 4 squares with total 64.

The initial terms of the *caraṇas* (rows) may be supposed to be (i) 7, 12, 17, 22 or (ii) 4, 11, 18, 25 or (iii) 1, 10, 19, 28, the common-difference being unity in each case. The corresponding squares are shown in Figs. 43-45.

7	15	22	20
23	19	8	14
10	12	25	17
24	18	9	13

Total = 64

Fig. 43

4	14	25	21
26	20	5	13
7	11	28	18
27	19	6	12

Total = 64

Fig. 44

1	13	28	22
29	21	2	12
4	10	31	19
30	20	3	11

Total = 64

Fig. 45

Construction of Irregular Series

Method 1: Nārāyaṇa gives the following rule for the determination of irregular series to be used for filling a square with a give total:

"For the determination of the *caraṇas* ("rows") assume the first term and the common-difference optionally. First write down the initial term and then add to it successively the product of the common-difference and the number of cells in a row, and do so as many times as the number of rows less one. The series thus formed is the *mukhapaṅkti* ("the optionally assumed series of initial terms"). To the last term of this series add the first term together with the product of the

common-difference into the number of rows minus one, and multiply by half the number of rows: this is the *mukhaphala* ("the total corresponding to the assumed series"). The desired total minus the *mukhaphala* is the *kṣepaphala* ("the total for the numbers to be interpolated"). Now determine the first term and the common-difference of a series in A.P. whose number of terms is equal to the number of rows and whose sum is equal to the *kṣepaphala*. Add the successive terms of the series thus obtained to the corresponding terms of the *mukhapaṅkti* ("the optionally assumed series of initial terms"). Thus will be determined the *caraṇas* for all magic squares"[25].

Example from Nārāyaṇa:

(i) Determine the *caraṇas* for a 4×4 magic square with total 40.

Optionally assume a series whose first term is 1 and the common-difference is 1. When the terms are placed in rows of four, the initial terms of the successive rows (i.e., *mukhapaṅkti*) are:

1, 5, 9, 13.

Since the number of *caraṇas* is 4,

$$\textit{mukhaphala} = \frac{4}{2}[13+1 + (4-1).1] = 34$$

$$\textit{kṣepaphala} = 40 - 34 = 6$$

Now, if A be the first term, and D, the common-difference and 6 the sum of an A.P. of 4 terms, we must have

$$\frac{6 - \frac{4}{2}(4-1)\,D}{4} = A$$

$\therefore$ A = 0, −3,.....

and D = 1, 3, ...

For the solution (A = 0, D = 1), the series is: 0, 1, 2, 3.

For the solution (A = −3, D = 3), the series is: −3, 0, 3, 6.

Therefore, the initial terms of the required *caraṇas* ("rows") are (1, 6, 11, 16) or (−2, 5, 12, 19).

(ii) Determine the *caraṇas* for the 4×4 square whose total is 64.

In this case, the *kṣepaphala* is $64 - 34 = 30$

so that $\frac{30 - 6D}{4} = A$

i.e. $A = 6, 3, 0, \ldots$

and $D = 1, 3, 5, \ldots$.

For the first solution,the series is (6, 7, 8, 9), for the second (3, 6, 9, 12) and for the third (0, 5, 10, 15). Therefore, the initial terms of the *caraṇas* are (7, 12, 17, 22) or (4, 11, 18, 25) or (1, 10, 19, 28).

The squares may now be constructed by the method of the knight's move.

Method 2[26]*:* "Divide the *kṣepaphala* ("total of numbers to be interpolated") by the *caraṇa* ("number of cells in a row"). The quotient increased by unity becomes the "*gaccha*"[27], provided the remainder is zero or equal to half the *caraṇa*. If the remainder is otherwise, the magic square is not possible. Add to the first and the second halves of the *mukhapaṅkti* respectively zero and half the *kṣepaphala* or these increased and decreased by unity successively. Thus will be determined the initial terms of the *caraṇas* in the cases of *samagarbha* and *viṣamagarbha* squares.

Examples from Nārāyaṇa

(i) To construct a 4×4 square with total 40.

Assuming the series of natural numbers, the *kṣepaphala* is $40 - 34 = 6$. This divided by the *caraṇa*, i.e. $6 \div 4$, gives the quotient 1 and remainder 2. The construction of the square is thus possible, and $1 + 1 = 2$ squares may be obtained. The *mukhapaṅkti* is 1, 5, 9, 13. Half of the *kṣepaphala* $= 3$

The numbers to be interpolated are, therefore, 0 and 3, or adding and subtracting unity, 1 and 2. Thus, adding these to the respective halves of the *mukhapaṅkti*, we get:

0	3	1	5	12	16
1	2	2	6	11	15

Interpolators — Initial terms of the rows[28]

Thus, the numbers to be filled in the square are:

1, 2, 3, 4	or	2, 3, 4, 5
5, 6, 7, 8		6, 7, 8, 9
12, 13, 14, 15		11, 12, 13, 14
16, 17, 18, 19		15, 16, 17, 18

and the corresponding squares are as shown in Figs. 46 and 47.

1	8	16	15
17	14	2	7
4	5	19	12
18	13	3	6

Total = 40

Fig. 46

2	9	15	14
16	13	3	8
5	6	18	11
17	12	4	7

Total = 40

Fig. 47

(ii) To construct a 4×4 square with total 64.

Here, as before the *kṣepaphala* $= 64 - 34 = 30$.

This divided by the *caraṇa*, i.e., $30 \div 4$ gives the quotient 7 and remainder 2. Thus, the square is possible and $7 + 1 = 8$ different squares may be obtained.

As before, the *mukhapaṅkti* is 1, 5, 9, 13. Half the *kṣepaphala* is 15. The numbers to be interpolated are 0, 15, or adding and subtracting unity successively to get 8 different pairs we have:

0	15
1	14
2	13
3	12
4	11
5	10
6	9
7	8

Adding these pairs to the respective halves of the *mukhapaṅkti* (1, 5, 9, 13), we get the following 8 sets for the initial terms of the rows:

1	5	24	28
2	6	23	27
3	7	22	26
4	8	21	25
5	9	20	24
6	10	19	23
7	11	18	22
8	12	17	21

Eight squares may now be constructed as before.

CHANGE OF SQUARES

"Construct a magic square of the type desired. Subtract its total from the given total, and divide by the number of cells in a line. On adding the quotient to the numbers in the cells of that square will be obtained the required square"[29].

Thus, to transform the 4×4 magic square of Fig. 22, with total 34, into another with total 100, one has simply to add $(100-34)/4$, i.e., 33/2 to the numbers in the cells of that magic square.

CONSTRUCTION BY SUPERPOSITION

"Construct two *samagarbha* squares, one called *chādaka* ("covering one") and the other called *chādya* ("one to be covered"). The superposition is to be made in the manner of folding the palms of the hands. Form a series with an optional first term and an optional common-difference and with as many terms as the "number"[30] of the square; this is the *mūlapaṅkti* ("basic series"). With another first term and common-difference form another series: this is called *parapaṅkti*. Multiplying the terms of the *parapaṅkti* by the quotient obtained on dividing the given total minus the sum of the *mūlapaṅkti* by the sum of the *parapaṅkti*, is produced the progression which is called *guṇapaṅkti* ("product-series"). Divide the *mūlapaṅkti* and the *guṇapaṅkti* by turning each upon itself, so that each part will have terms equivalent to half the "number" of the square. The numbers are written down vertically, one above the other, and directly in the *chādaka* ("covering one") and in another fashion (i.e., horizontally and inversely) in the *chādya* ("one to be covered"). In the first, fill thus successively half the rows and in the second half the columns[31]: Fill the other half of each square in the contrary way. This method of constructing magic squares by superposition is taught by the son of Nṛhari (i.e., by Nārāyaṇa)"[32].

Examples from Nārāyaṇa

(i) To construct a 4×4 magic square with total 40.

Assume the *mūlapaṅkti* to be 1, 2, 3, 4

Let the *parapaṅkti*[33] be 0, 1, 2, 3

The multiplier $= (40 - 10)/6 = 5$

$\therefore$ The *gunapaṅkti* is 0, 5, 10, 15.

Writing the *mūlapaṅkti* and *guṇapaṅkti* by turning them upon themselves, we get

1	2		0	5	
		and			respectively.
4	3		15	10	

Taking the first set, placing it vertically and then filling with it the horizontal half of a 4×4 square, we get fig. 48.

2	3	2	3
1	4	1	4

Fig. 48

Then filling the other half with the same numbers in the inverse order, we get the *chādaka* (Fig. 49).

(A)

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

Chādyaka

Fig. 49

In the same way, filling horizontally with the second set, we get the *chādya* (Fig. 50).

(B)

5	0	10	15
10	15	5	0
5	0	10	15
10	15	5	0

Chādya

Fig. 50

Then, folding (A) over (B) and adding the numbers, we get the required square with total 40 (Fig. 51).

(AB)

8	2	13	17
14	16	9	1
7	3	12	18
11	19	6	4

Total = 40

Aliter. Or, if we take the *mūlapaṅkti* as before but the *parapaṅkti* as 1, 2, 3, 4, the multiplier is (40 − 10)/10 = 3, so that the *guṇapaṅkti* is 3, 6, 9, 12. Thus we get

1	2		3	6
		and		
4	3		12	9

and the corresponding squares are

(A′)

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

Fig. 52

(B′)

6	3	9	12
9	12	6	3
6	3	9	12
9	12	6	3

Fig. 53

(A′B′)

9	5	12	14
13	13	10	4
8	6	11	15
10	16	7	7

Total = 40

Fig 54 (vide ref. 34)

(ii) To construct a 4 × 4 square with total 64.

Here, taking

the *mūlapaṅkti* as 1, 2, 3, 4

and the *parapaṅkti* as 0, 1, 2, 3

the multiplying factor is $\dfrac{64-(1+2+3+4)}{(0+1+2+3)} = 9$

∴ the *guṇapankti* is 0, 9, 18, 27.

Arranging, we have correspondingly

1	2	0	9
4	3	27	18

(A)

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

Fig. 55

(B)

9	0	18	27
18	27	9	0
9	0	18	27
18	27	9	0

Fig. 56

(AB)

12	2	21	29
22	28	13	1
11	3	20	30
19	31	10	4

Total = 64

Fig. 57

as before (Figs. 55-57).

The above method of constructing squares was rediscovered in Europe by M. de la Hire (1705), and is now attributed to him.

(iii) to construct a 8×8 square with total 260.

Let the *mūlapaṅkti* be 1, 2, 3, 4, 5, 6, 7, 8,

and the *parapaṅkti* 0, 1, 2, 3, 4, 5, 6, 7.

$$\text{The multiplying factor} = \frac{260 - \frac{1}{2} \cdot 8\,(8+1)}{\frac{1}{2} \cdot 7\,(7+1)} = 8$$

The *guṇapaṅkti* is 0, 8, 16, 24, 32, 40, 48, 56.

Breaking up the *mūlapaṅkti* and *guṇpaṅ* ˙ into halves and writing them by turning upon themselves we have

(a) (b)

1	2	3	4		0	8	16	24
				and				
8	7	6	5		56	48	40	32

Hence, the preliminary squares are as shown in Figs. 58 and 59.

4	5	4	5	4	5	4	5
3	6	3	6	3	6	3	6
2	7	2	7	2	7	2	7
1	8	1	8	1	8	1	8
5	4	5	4	5	4	5	4
6	3	6	3	6	3	6	3
7	2	7	2	7	2	7	2
8	1	8	1	8	1	8	1

Fig. 58

24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0

Fig. 59

Superposing these two as in the hinge, we get Fig. 60.

60	53	44	37	4	13	20	29
3	14	19	30	59	54	43	38
58	55	42	39	2	15	18	31
1	16	17	32	57	56	41	40
61	52	45	36	5	12	21	28
6	11	22	27	62	51	46	35
63	50	47	34	7	10	23	26
8	9	24	25	64	49	48	33

Total = 260

Fig. 60 (vide ref. 35)

Second Method: "In as many 4 × 4 squares as are present in the *samagarbha* (4n × 4n square) such as 8 × 8 square, etc., write the numbers produced in the series, as in the method of the 4 × 4 square by right and left (knight's) moves. Thus is said the easy method of constructing *samagarbha* (4n × 4n) squares such as 8 × 8 square, etc.[36].

Example from Nārāyaṇa

To construct a 8 × 8 square with total 260.

It is easily seen that the series of natural numbers from 1 to 64 is to be used. Writing the numbers 1 to 64 in groups of 4, we have

	1	8	9	16	48	41	40	33	
	2	7	10	15	47	42	39	34	
(I)	3	6	11	14	46	43	38	35	(III)
	4	5	12	13	45	44	37	36	
	32	25	24	17	49	56	57	64	
(II)	31	26	23	18	50	55	58	63	(IV)
	30	27	22	19	51	54	59	62	
	29	28	21	20	52	53	60	61	

Interchanging the figures in the third and fourth columns, as in the method of filling 4×4 squares, we get

	1	8	16	9	48	41	33	40	
I	2	7	15	10	47	42	34	39	III
	3	6	14	11	46	43	35	30	
	4	5	13	12	45	44	36	37	
	32	25	17	24	49	56	64	57	
	31	26	18	23	50	55	63	58	IV
II	30	27	19	22	51	54	62	59	
	29	28	20	21	52	53	61	60	

Taking the first rows of I and II to fill the first 4×4 square, the second rows to fill the second and so on, we get Fig. 61.

1	32			2	31		
		8	25			7	26
16	17			15	18		
		9	24			10	23
4	29			3	30		
		5	28			6	27
13	20			14	19		
		12	21			11	22

Fig. 61

Then taking the first rows of III and IV to fill the remaining cells of the first 4×4 square, the second rows to fill the remaining cells of the second 4×4 square and so on, we get Fig. 62.

1	32	49	48	2	31	50	47
56	41	8	25	55	42	7	26
16	17	64	33	15	18	63	34
57	40	9	24	58	39	10	23
4	29	52	45	3	30	51	46
53	44	5	28	54	43	6	27
13	20	61	36	14	19	62	35
60	37	12	21	59	38	11	22

Total = 260
Fig 62 (vide ref. 37)

Viṣamagarbha Squares

Nārāyaṇa gives two methods of construction of the (4n+2) × (4n+2) squares.

First method: This method is described by Nārāyaṇa thus:

"The measure of the *śliṣṭa*[38] cells is half of half the "number of the square" minus one. All over the square write down the numbers in connected cells in the direct and inverse order, one below the other (in rows). The numbers standing in the middle two columns above and below the middle two rows, excepting those in the last but one column below, should be interchanged (by one place anticlockwise turning). Then the two middle numbers in the extreme right of the right half of the square should be interchanged with the corresponding ones of the left half of the square, which lie attached to the diagonal. Finally the numbers of the *śliṣṭa* cells in the upper and lower halves of the square should be interchanged symmetrically. Such is the procedure of filling the cells with numbers by the method of *śliṣṭa* cells. The numbers in the cells attached to the diagonal in the right half of the square should be left as they are. Others may be interchanged if necessary to make up the total. This is the method of constructing the *Viṣamagarbha-bhadra* taught by Nārāyaṇa"[39].

Examples from Nārāyaṇa

(i) To construct a 6×6 square with total 111.

It is easily seen that the series of natural numbers from 1 to 36 is to be used.

The measure of the *śliṣṭa* cells = (3 − 1)/2 = 1.

The numbers 1 to 36 are placed in the 6 × 6 square in the direct and inverse order, as in Fig. 63.

1	2*	3	4	5*	6
12*	11	10	9	8	7*
13*	14	15	16	17	18*
24*	23	22	21	20	19*
25*	26	27	28	29	30*
36	35*	34	33	32*	31

Fig. 63

There is only one *śliṣṭa* cell in each half row. These lie at the ends and are marked by asterisks.

The numbers in the two middle columns lying above and below the two middle rows excepting those in the last but one co-column below are interchanged (by one place anticlockwise turning) as shown below. The numbers in the extreme right cells of the two middle rows are interchanged with the corresponding ones of the left half of the square. This gives Figs. 64 and 65.

1	*	4	33	*	6
*	11	9	28	8	*
*	14	15	16	17	18*
*	23	22	21	20	19*
*	26	27	10	29	*
36	*	34	3	*	31

Fig. 64

1	*	4	33	*	6
*	11	9	28	8	*
*	14	18	16	17	15*
*	23	19	21	20	22*
*	26	27	10	29	*
36	*	34	3	*	31

Fig. 65

The numbers standing in the *ślista* cells above are then interchanged with the corresponding ones below, giving Fig. 66.

1	35	4	33	32	6
25	11	9	28	8	30
24	14	18	16	17	22
13	23	19	21	20	15
12	26	27	10	29	7
36	2	34	3	5	31

Total = 111

Fig. 66

(ii) To fill a 10×10 square with the natural numbers 1 to 100.

In this case, the total $= \dfrac{100 \times 101}{2 \times 10} = 505.$

Placing the numbers 1 to 100 in a 10×10 square, we get Fig. 67.

1	2	3	4	5	6	7	8	9	10
20	19	18	17	16	15	14	13	12	11
21	22	23	24	25	26	27	28	29	30
40	39	38	37	36	35	34	33	32	31
41	42	43	44	45	46	47	48	49	50
60	59	58	57	56	55	54	53	52	51
61	62	63	64	65	66	67	68	69	70
80	79	78	77	76	75	74	73	72	71
81	82	83	84	85	86	87	88	89	90
100	99	98	97	96	95	94	93	92	91

Fig. 67

Interchanging the numbers in the middle columns as directed and also those in the extreme right cells of the two middle rows with the corresponding ones of the left half of the square, we get Fig. 68.

1	*	*	4	6	95	7	*	*	10
*	19	*	17	15	86	14	*	12	*
*	*	23	24	26	75	27	28	*	*
*	*	38	37	35	66	34	33	*	*
*	*	43	44	50	46	47	48	*	* 45
*	*	58	57	51	55	54	53	*	* 56
*	*	63	64	65	36	67	68	*	*
*	*	78	77	76	25	74	73	*	*
*	82	*	84	85	16	87	*	89	*
100	*	*	97	96	5	94	*	*	91

Fig. 68

Then interchanging the numbers in the *śliṣṭa* cells (marked by asterisks) as before we have the required square (Fig. 69).

1	99	98	4	6	95	7	93	92	10
81	19	83	17	15	86	14	88	12	90
80	79	23	24	26	75	27	28	72	71
61	62	38	37	35	66	34	33	69	70
60	59	43	44	50	46	47	48	52	56
41	42	58	57	51	55	54	53	49	45
40	39	63	64	65	36	67	68	32	31
21	22	78	77	76	25	74	73	29	30
20	82	18	84	85	16	87	13	89	11
100	2	3	97	96	5	94	8	9	91

Total = 505

Fig. 69

(iii) To construct a 14 × 14 square with the series of natural numbers.

The numbers filled continuously in a 14 × 14 square and then interchanged according to Nārāyaṇa's rule give Figs. 70 and 71.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
28	27	26	25	24	23	22	21	20	19	18	17	16	15
29	30	31	32	33	34	35	36	37	38	39	40	41	42
56	55	54	53	52	51	50	49	48	47	46	45	44	43
57	58	59	60	61	62	63	64	65	66	67	68	69	70
84	83	82	81	80	79	78	77	76	75	74	73	72	71
85	86	87	88	89	90	91	92	93	94	95	96	97	98
112	111	110	109	108	107	106	105	104	103	102	101	100	99
113	114	115	116	117	118	119	120	121	122	123	124	125	126
140	139	138	137	136	135	134	133	132	131	130	129	128	127
141	142	143	144	145	146	147	148	149	150	151	152	153	154
168	167	166	165	164	163	162	161	160	159	158	157	156	155
169	170	171	172	173	174	175	176	177	178	179	180	181	182
196	195	194	193	192	191	190	189	188	187	186	185	184	183

Key-square

Fig. 70

1	195	194	193	5	6	8	189	9	10	186	185	184	14
169	27	171	172	24	23	21	176	20	19	179	180	16	182
168	167	31	165	33	34	36	161	37	38	158	40	156	155
141	142	143	53	52	51	49	148	48	47	46	152	153	154
140	139	138	60	61	62	64	133	65	66	67	129	128	127
113	114	115	81	80	79	77	120	76	75	74	124	125	126
112	111	110	88	89	90	98	92	93	94	95	101	100	106
85	86	87	109	108	107	99	105	104	103	102	96	97	91
84	83	82	116	117	118	119	78	121	122	123	73	72	71
57	58	59	137	136	135	134	63	132	131	130	68	69	70
56	55	54	144	145	146	147	50	149	150	151	45	44	43
29	30	166	32	164	163	162	35	160	159	39	157	41	42
28	170	26	25	173	174	175	22	177	178	18	17	181	15
196	2	3	4	192	191	190	7	188	187	11	12	13	183

Total = 1379

Fig. 71

Remarks

It will be observed that when the series employed is in A.P., the squares are constructed by making the minimum interchanges expressly stated by Nārāyaṇa. That this is so in all cases is illustrated by the 18×18 squares constructed according to this method (Figs. 72 and 73).

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
36	35	34	33	32	31	30	29	28	27	26	25	24	23	22	21	20	19
37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
72	71	70	69	68	67	66	65	64	63	62	61	60	59	58	57	56	55
73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
108	107	106	105	104	103	102	101	100	99	98	97	96	95	94	93	92	91
109	110	111	112	113	114	115	116	117	118	119	120	121	122	123	124	125	126
144	143	142	141	140	139	138	137	136	135	134	133	132	131	130	129	128	127
145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160	161	162
180	179	178	177	176	175	174	173	172	171	170	169	168	167	166	165	164	163
181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198
216	215	214	213	212	211	210	209	208	207	206	205	204	203	202	201	200	199
217	218	219	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234
252	251	250	249	248	247	246	245	244	243	242	241	240	239	238	237	236	235
253	254	255	256	257	258	259	260	261	262	263	264	265	266	267	268	269	270
288	287	286	285	284	283	282	281	280	279	278	277	276	275	274	273	272	271
289	290	291	292	293	294	295	296	297	298	299	300	301	302	303	304	305	306
324	323	322	321	320	319	318	317	316	315	314	313	312	311	310	309	308	307

Key-square

Fig. 72

When, however, the series to be used is a broken series, other changes have to be made. For instance, to construct a 6×6 square with total 132, one may take the series of initial terms 2, 9, 16, 23, 30, 37 and common-difference 1. Proceeding according to the rule, we get Fig. 74.

But in this square, the third and fourth rows do not have the desired total. We, therefore, replace the initial numbers 28 and 16 of the third and fourth rows by 29 and 15 respectively and thus we get the magic square shown in Fig. 75.

1	323	322	321	320	6	7	8	10	315	11	12	13	311	310	309	308	18
289	35	291	292	293	31	30	29	27	298	26	25	24	302	303	304	20	306
288	287	39	285	284	42	43	44	46	279	47	48	49	275	274	52	272	271
253	254	255	69	257	67	66	65	63	262	62	61	60	266	58	268	269	270
252	251	250	249	77	78	79	80	82	243	83	84	85	86	238	237	236	235
217	218	219	220	104	103	102	101	99	226	98	97	96	95	231	232	233	234
216	215	214	213	113	114	115	116	118	207	119	120	121	122	202	201	200	199
181	182	183	184	140	139	138	137	135	190	134	133	132	131	195	196	197	198
180	179	178	177	149	150	151	152	162	154	155	156	157	158	166	165	164	172
145	146	147	148	176	175	174	173	163	171	170	169	168	167	159	160	161	153
144	143	142	141	185	186	187	188	189	136	191	192	193	194	130	129	128	127
109	110	111	112	212	211	210	209	208	117	206	205	204	203	123	124	125	126
108	107	106	105	221	222	223	224	225	100	227	228	229	230	94	93	92	91
73	74	75	76	248	247	246	245	244	81	242	241	240	239	87	88	89	90
72	71	70	256	68	258	259	260	261	64	263	264	265	59	267	57	56	55
37	38	286	40	41	283	282	281	280	45	278	277	276	50	51	273	53	54
36	290	34	33	32	294	295	296	297	28	299	300	301	23	22	21	305	19
324	2	3	4	5	319	318	317	316	9	314	313	312	14	15	16	17	307

Total = 2925

Fig. 73

In this magic square, no number has been repeated. Replacement of the numbers 17 and 27 (standing in the third and fourth rows) by 18 and 26, or 20 and 24 by 21 and 23, or 26 and 18 by 27 and 17 will also yield magic squares with total 132, but there will be repetitions of two numbers.

Nārāyaṇa, however, gives Fig. 74 as a 6×6 magic square with total 132. But as pointed out above, it is truly speaking not a magic square.

2	41	5	39	38	7
30	13	11	33	10	35
28	17	21	19	20	26
16	27	23	25	24	18
14	31	32	12	34	9
42	3	40	4	6	37

Fig. 74

2	41	5	39	38	7
30	13	11	33	10	35
29	17	21	19	20	26
15	27	23	25	24	18
14	31	32	12	34	9
42	3	40	4	6	37

Total = 132

Fig. 75

If we use the series of initial terms 1, 7, 13, 26, 32, 38 and common-difference 1, and proceed as above, we shall get Fig. 76.

1	42	4	40	39	6
32	11	9	35	8	37
31	14	18	16	17	29
13	30	26	28	27	15
12	33	34	10	36	7
43	2	41	3	5	38

Fig. 76

Here also, the third and fourth rows do not have the desired total. But if we replace the numbers 31 and 13, in those rows, by 38 and 6, or 14 and 30 by 21 and 23, or 17 and 27 by 24 and 20, or 29 and 15 by 36 and 8 respectively, we shall get 4 magic squares with total 132. In two of these magic squares there will be no repetition of numbers, but in the other two there will be repetition of numbers.

Second method: "In the *Viṣamagarbha* squares such as 6×6, etc. the two middle lines of cells (both horizontal and vertical) are called *Pīṭha*. Fill the cells of the square with the numbers (of the given or chosen series) in the direct order. Reverse the number in the cells of each diagonal. Then interchange the numbers lying at the north-east corner between the diagonal and the *Pīṭha* with the numbers in the (corresponding) opposite cells. Then interchange the two numbers at the south *Pīṭha*, and also those at the west *Pīṭha*. Thus will be obtained the desired total in the horizontal and vertical outskirts of the square. The interchange of the numbers in the other cells should be made as required in order to make up the total by noting the deficit or excess from it"[40].

Examples (i): To construct a 6×6 square with the series of natural numbers.

The numbers are placed in the square in the direct order as in Fig. 77.

In the above, the *Pīṭhas* ("central rows and columns") are marked by thick lines. The directions are indicated by the letters E, N, W and S. The numbers in the diagonal cells are reversed. The numbers 2 and 7 lying at the north-east corner between the diagonal cells and the *Pīṭha* cells are interchanged with the numbers 32 and 12, respectively, which lie in the corresponding opposite cells. Then the numbers 18 and 24 at the south *Pīṭha* are interchanged; so also are interchanged the numbers 33 and 34 at the west *Pīṭha*. Thus, we have Fig. 78,

E

N (left), S (right), W (bottom)

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

W

Fig. 77

36	32	3	4	5	31
12	29			26	7
13		22	21		24
19		16	15		18
25	11			8	30
6	2	34	33	35	1

Fig. 78

in which the totals of the bounding rows and columns are as desired. The sums of the diagonal cells are also as desired. The other numbers should now be interchanged by trial to get the desired total 111. The squares shown in Figs.79 and 80 result.

36	32	3	4	5	31
12	29	27	10	26	7
13	17	22	21	14	24
19	20	16	15	23	18
25	11	9	28	8	30
6	2	34	33	35	1

Total = 111

Fig. 79

36	32	3	4	5	31
12	29	9	28	26	7
13	14	22	21	17	24
19	23	16	15	20	18
25	11	27	10	8	30
6	2	34	33	35	1

Total = 111

Fig. 80

(ii) To construct a 10 × 10 square with the series of natural numbers.

The above process gives the square shown in Fig. 81.

100	92	93	94	5	6	7	8	9	91
20	89	88	14	16	15	87	83	82	11
30	29	78	77	75	26	74	73	22	21
40	39	63	67	65	66	64	38	32	31
41	49	48	54	56	55	57	43	42	60
51	52	53	47	46	45	44	58	59	50
61	62	33	37	35	36	34	68	69	70
71	72	28	27	25	76	24	23	79	80
81	19	18	84	86	85	17	13	12	90
10	2	3	4	96	95	97	98	99	1

Fig. 81

VIṢAMA SQUARES

Nārāyaṇa gives two methods for the construction of the "*Viṣamabhadra* ("Odd squares"). The first of these is Nārāyaṇa's own method, the method of superposition, which was rediscovered in the west by M de la Hire (1705). The second method seems to have been known in India before Nārāyaṇa.

First method: "Determine the *mūlapaṅkti* and *guṇapaṅkti* in the way indicated before. The first term of the former should be placed in the centre cell of the top row of the first of the (*chādya* and *chādaka*) squares. Beneath it should be written down vertically the successive terms of the series. The other columns should be filled similarly, so that the numbers in the top row are in order. In the same way, beginning with the first term of the second series fill up the second square. The method of superposition of the *chādya* ("one to be covered") and *chādaka* ("covering one") is as before"[41].

Examples from Nārāyaṇa

(i) To construct a 3×3 square with total 24.

Assume the *mūlapaṅkti* ("basic series") to be 1, 2, 3.

Let the *parapaṅkti* ("second series") be 0, 1, 2.

Then the multiplying factor is $\dfrac{24-(1+2+3)}{(0+1+2)} = 6$

Therefore, the *guṇapaṅkti* is 0, 6, 12.

Now, filling the *chādaka* ("one to be covered") square with the *mūlapaṅkti* and *chādaka* ("covering one") with the *guṇapaṅkti* as directed in the rule, we get Figs. 82 and 83.

Superposing these as in a hinge, we have the required square (Fig. 84).

3	1	2
1	2	3
2	3	1

Fig. 82

12	0	6
0	6	12
6	12	0

Fig. 83

9	1	14
13	8	3
2	15	7

Total = 24

Fig. 84

(ii) To construct a 5×5 square with total 90.

Let the *mūlapaṅkti* be 1, 2, 3, 4, 5

Also let the *parapaṅkti* be 1, 2, 3, 4, 5

Then the multiplier is $\frac{90-(1+2+3+4+5)}{1+2+3+4+5} = 5$

Therefore, the *guṇapaṅkti* is 5, 10, 15, 20, 25.

Filling the squares as before, we have Figs. 85-87.

4	5	1	2	3
5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2

Chādya
Fig. 85

20	25	5	10	15
25	5	10	15	20
5	10	15	20	25
10	15	20	25	5
15	20	25	5	10

Chādaka
Fig. 86

19	15	6	27	23
25	16	12	8	29
26	22	18	14	10
7	28	24	20	11
13	9	30	21	17

Total = 90
Fig. 87

5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3

Chādya

(iii) To construct a 7×7 square with total 238.

Here, taking the *mūlapaṅkti* as 1, 2, 3, 4, 5, 6, 7, and the *parapaṅkti* as 0, 1, 2, 3, 4, 5, 6, the *guṇapaṅkti* is 0, 10, 20, 30, 40, 50, 60. The squares obtained as above are shown in Figs. 88-90.

40	50	60	0	10	20	30
50	60	0	10	20	30	40
60	0	10	20	30	40	50
0	10	20	30	40	50	60
10	20	30	40	50	60	0
20	30	40	50	60	0	10
30	40	50	60	0	10	20

Chādaka

Fig. 89

35	26	17	1	62	53	44
46	37	21	12	3	64	55
57	41	32	23	14	5	66
61	52	43	34	25	16	7
2	63	54	45	36	27	11
13	4	65	56	47	31	22
24	15	6	67	51	42	33

Total = 238

Fig. 90

Second method: "In the first cell of a middle line (of cells) write the first term of the series of numbers, and in the cell beside the opposite cell of the same line (write) the next number. Then, in the cells lying along the shorter diagonal from that write the following numbers. (On reaching an extremity) continue the filling beginning with the cell of the opposite line which will be diagonally in front (considering the square to be rolled on a cylinder). When the next diagonal cell is found to be already filled up, begin from the cell behind and fill successively (in the same way). In the *viṣamabhadra* there will be eight varieties"[42].

Examples from Nārāyaṇa

(i) To construct a 3×3 square with the series of natural numbers.

Writing 1 at the top of the middle line (column), the 2 in the last cell of the next column and proceeding diagonally upwards we have Fig. 91.

8	1	6
3	5	7
4	9	2

Total = 15

Fig. 91

In the above, whenever a block occurs, we begin with the cell underneath.

Another filling would be as per Fig. 92.

As the filling can be started by placing the first term in any one of the four centre cells of the outskirts, there will be altogether 8 different squares, as stated by Nārāyaṇa.

(ii) To construct a 3×3 square with total 24.

Nārāyaṇa uses an irregular series for filling up the square. According to the method for finding out such series, we get 3, 7, 11 as the initial terms of the *caraṇas*, the common difference being 1. The numbers to be filled are, therefore,

3, 4, 5
7, 8, 9
11, 12, 13

Hence the magic square is as in Fig. 93.

6	1	8
7	5	3
2	9	4

Total = 15
Fig. 92

7	5	12
13	8	3
4	11	9

Total = 24
Fig. 93

Note: In this and the following squares, the filling begins from the extreme right cell of the middle row.

(iii) To construct a 5×5 square with total 90.

Here, the initial terms of the *caraṇas* are found to be 4, 10, 16, 22, 28, the common-difference being unity. The square is shown in Fig. 94.

(iv) To construct a 7×7 square with total 238.

In this case, the initial terms of the *caraṇas* may be taken as 7, 15, 23, 31, 39, 47, 55, the common-difference being unity. The square is shown in Fig. 95.

16	14	7	30	23
24	17	10	8	31
32	25	18	11	4
5	28	26	19	12
13	6	29	22	20

Total = 90
Fig. 94

31	29	20	11	58	49	40
41	32	23	21	12	59	50
51	42	33	24	15	13	60
61	52	43	34	25	16	7
8	55	53	44	35	26	17
18	9	56	47	45	36	27
28	19	10	57	48	39	37

Total = 238
Fig. 95

The magic squares constructed by the above method are such that the sum of any two numbers that are geometrically equidistant from the centre is equal to twice the centre number. Such squares are called perfect by W.S. Andrews[43].

OTHER MAGIC FIGURES

Nārāyaṇa says: "With the help of 4×4 magic squares filled by natural numbers 1, etc. construct a magic rectangle or 4n×4n magic square. From it one can always construct other magic figures. Lines drawn through the corners in any desired way so as always to keep the number of cells the same give rise to the figures of *Vitāna* ("canopy"), *Maṇḍapa* ("altar"), *Vajra* ("diamond"), etc. Those are *Saṅkīrṇabhadra* ("other magic figures"). By the meeting together of lines between two cells and two diagonals are produced bases and uprights of triangle-pairs in all directions. Here, the triangles are filled with the numbers of a magic rectangle produced by 4n×4n squares, first in the direct order and then in the inverse order and so on. Such is the method of filling magic figures"[44].

Besides the three types of magic figures mentioned above, Nārāyaṇa has given rules for the construction of many other types of figures with illustration. These figures will be given and their peculiarities pointed out. The rules regarding their constructions will not be given, as they are apparent from the figures.

Vitāna ("canopy"): The figure is as shown in Fig. 96.

1	16	25	24	2	15	26	23
28	21	4	13	27	22	3	14
8	9	32	17	7	10	31	18
29	20	5	12	30	19	6	11

Vitāna or Canopy
Fig. 96

This is a rectangle constructed with the natural numbers 1 to 32 and consists of two 4×4 squares. The numbers are filled according to the method of 4n×4n squares given before[45]. It will be observed that the total of each row in the above is 132 and that of each column is 66.

For another magic rectangle constructed with the natural numbers 1 to 48 and consisting of three squares, see below (Fig. 105)

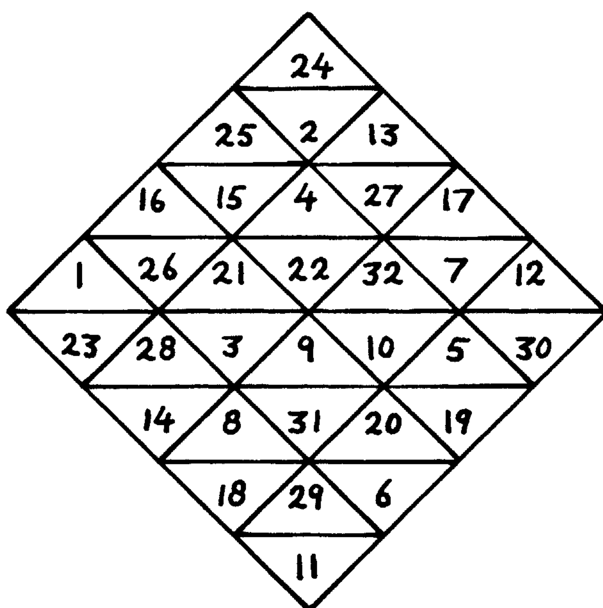
Maṇḍapa ("Altar"): The figure is

1 / 23	16 / 26	25 / 15	24 / 2	1 / 23	16 / 26	25 / 15	24 / 2
14 / 28	3 / 21	22 / 4	27 / 13	14 / 28	3 / 21	22 / 4	2 / 13
8 / 18	9 / 31	32 / 10	17 / 7	8 / 18	9 / 31	32 / 10	17 / 7
11 / 29	6 / 20	19 / 5	30 / 12	11 / 29	6 / 30	19 / 5	3 / 12

Mandapa or Altar
Fig. 97

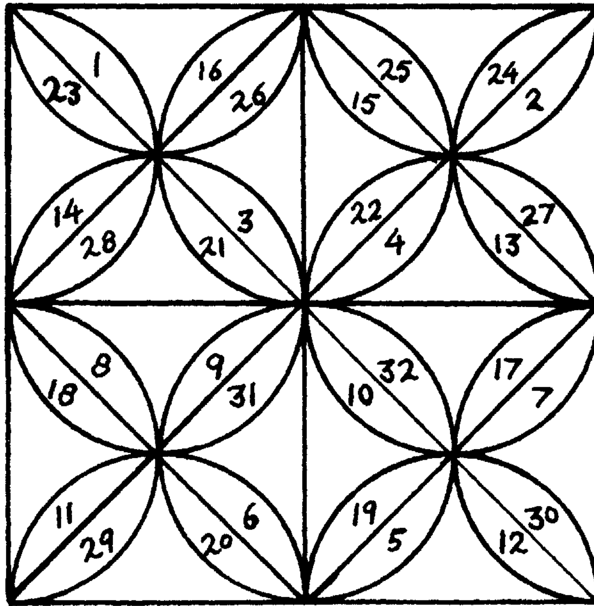
Here the numbers of the magic rectangle (Fig. 96) have been used by taking them successively in rows. Here, any set of eight numbers occurring together[46], horizontally, vertically or diagonally, gives the total 132. The eight numbers lying in a square have the same total 132. There is cylindrical symmetry, i.e., if the figure be rolled on a cylinder, any continuous eight numbers or those lying in a square give the total 132. It is easy to find 26 sets of eight numbers having the same total 132.

Vajra ("Diamond"): The figure (Fig. 98) is constructed from the magic rectangle in Fig. 96. Any eight numbers lying together in the same line, as well as the vertical diagonal, have the same sum 132. The sum of two horizontal rows, one in the upper half of the square and the other in the lower half, together containing eight numbers is 132. The sum of eight numbers lying in a small square is 132. In this case, it is easy to find 32 sets of eight numbers having the same total.



Vajra or Diamond
Fig. 98

Padma ("Lotus"): The figure (Fig. 99) is constructed from the rectangle in Fig. 96. Any set of eight numbers taken vertically, horizontally (along lines side by side) or in any four leaves symmetrically situated give the same total 132. There is cylindrical symmetry. In this case, 32 sets of eight numbers having the same total can be easily picked out.



Padma or Lotus
Fig. 99

Vajra ("Diamond"): The *vajra* ("Diamond") (Fig. 100) uses the numbers of the 8 × 8 magic square in Fig. 62.

46 / 1 32 / 51	27 / 49 48 / 6	35 / 2 31 / 62	22 / 50 47 / 11
30 / 56 41 / 3	43 / 8 25 / 54	19 / 55 42 / 14	38 / 7 26 / 59
45 / 16 17 / 52	28 / 64 33 / 5	36 / 15 18 / 61	21 / 63 34 / 12
29 / 57 40 / 4	44 / 9 24 / 53	20 / 58 39 / 13	37 / 10 23 / 60

Vajra or Diamond
Fig. 100

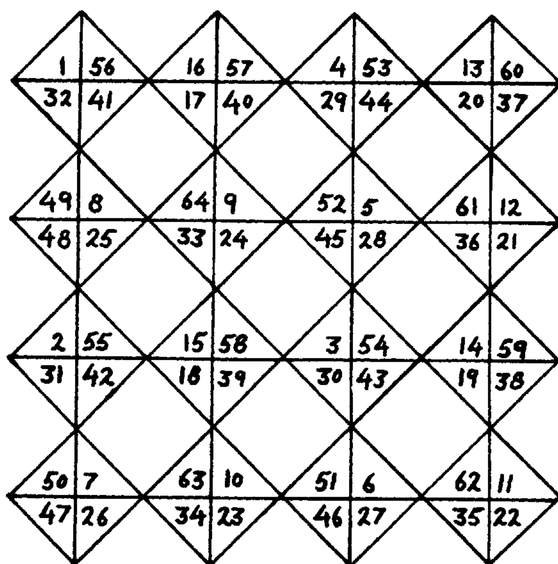
In the above groups of 16 numbers with the same total 520, groups of eight numbers with total 260 and also groups of four numbers with total 130 can be picked out easily. The sixteen numbers may be taken horizontally, vertically, and in two rings, etc. Groups of eight may be taken horizontally, vertically, diagonally, in rings, etc. Groups of four may be taken horizontally or vertically, as half rows or columns, in small squares, etc.

Maṇḍapa: The following *maṇḍapa* ("altar") is constructed by using the numbers of the 8 × 8 magic square in Fig. 62. It has groups of sixteen, eight and four numbers having equal totals, as in the *Vajra*.

1	32	49	48	2	31	50	47
46	51	30	3	45	52	29	4
27	6	43	54	28	5	44	53
56	41	8	25	55	42	7	26
16	17	64	33	15	18	63	34
35	62	19	14	36	61	20	13
22	11	38	59	21	12	37	60
57	40	9	24	58	39	10	23

Maṇḍapa
Fig. 101

Sarvatobhadra (*"Perfect magic figure"*): In this figure (Figs. 102 and 103) constructed from the 8 × 8 magic square in Fig.62, the totals of all four, eight and sixteen numbers are 130, 260 and 520 respectively. The figure is perfectly continuous.

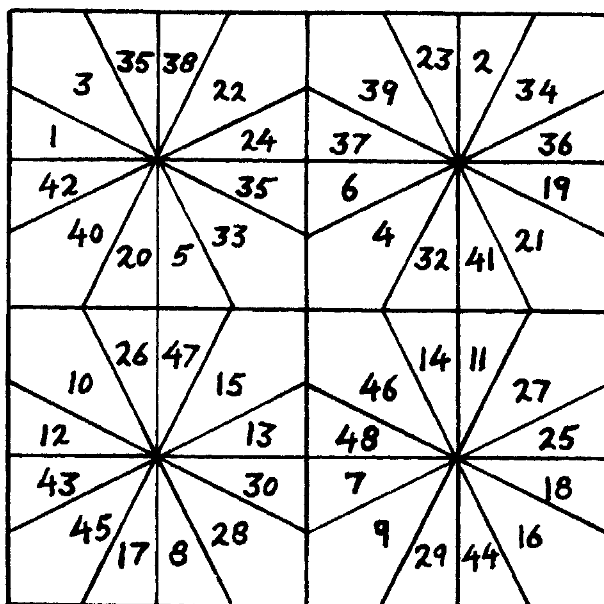


Sarvatobhadra
Fig. 102

1	56			16	57			4	53			13	60
32	41			17	40			29	44			20	37
49	8			64	9			52	5			61	12
48	25			33	24			45	28			36	21
2	55			15	58			3	54			14	59
31	42			18	39			30	43			19	38
50	7			63	10			51	6			62	11
47	26			34	23			47	27			35	22

Sarvatobhadra
Fig. 103

Dvādaśa-Kara ("Twelve-hands"): The figure (Fig. 104) is constructed by the numbers of the 12×4 rectangle (Fig. 105) using the numbers 1 to 48.



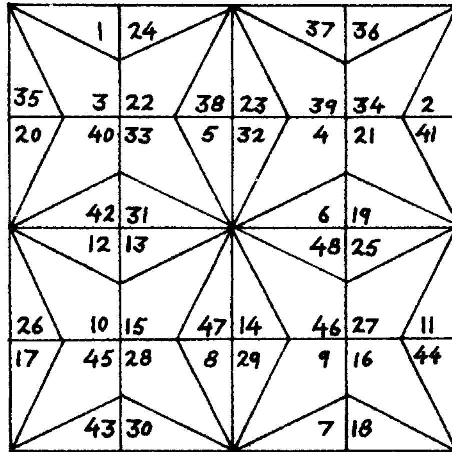
Dvādaśakara
Fig. 104

In the above figure all groups of 12, of 8 or of 4 numbers have equal totals, 294, 196 and 98 respectively.

1	24	37	36	2	23	38	35	3	22	39	34
42	31	6	19	41	32	5	20	40	33	4	21
12	13	48	25	11	14	47	26	10	15	46	27
43	30	7	18	44	29	8	17	45	28	9	16

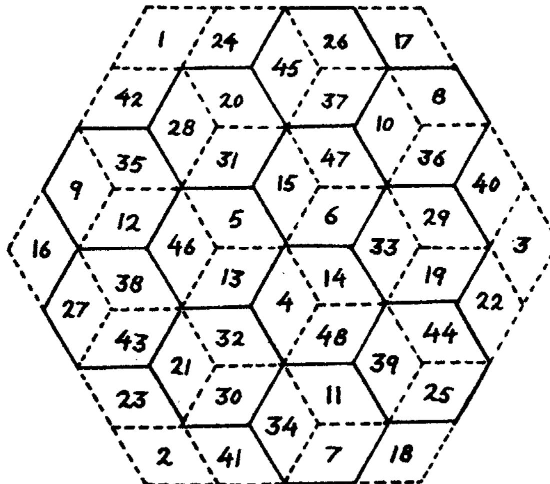
Fig. 105

Vajra Padma ("Diamond lotus"): The figure (Fig. 106) is constructed with the numbers of the 12×4 rectangle given above. In this figure, every group of four numbers whether occurring in a line or cells has the total 98, every group of eight numbers has the total 196 and every group of 12 numbers taken horizontally, vertically or in a circle has the total 294.



Vajra Padma
Fig. 106

Ṣaḍasra ("Hexagon"): The figure (Fig. 107) is constructed with the numbers of the 12×4 rectangle. In it every group of twelve numbers has the same sum 294.



Ṣaḍasra or Hexagon
Fig. 107

Padma Vṛtta ("Inscribed lotus"): The figure (Fig. 108) is constructed with the numbers of the 12 × 4 magic rectangle. Every group of twelve numbers has the same sum 294.

Padmavṛtta or Lotus circle

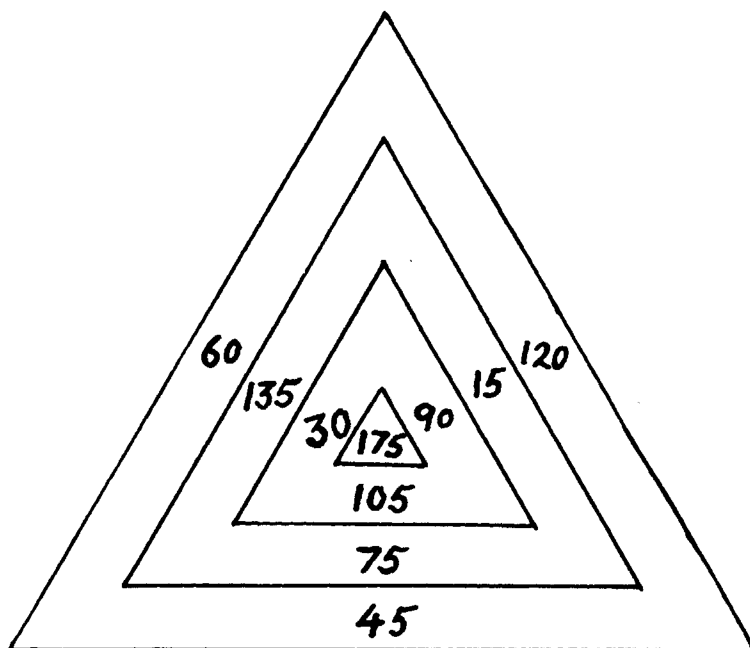
Fig. 108

Magic Triangle: Nārāyaṇa has proposed the problem of constructing a magic triangle with total 400. His magic triangle is constructed with the help of the numbers of a magic square whose total is 225.

120	15	90
45	75	105
60	135	30

Total = 225

Fig. 109



Total = 400
Fig. 110

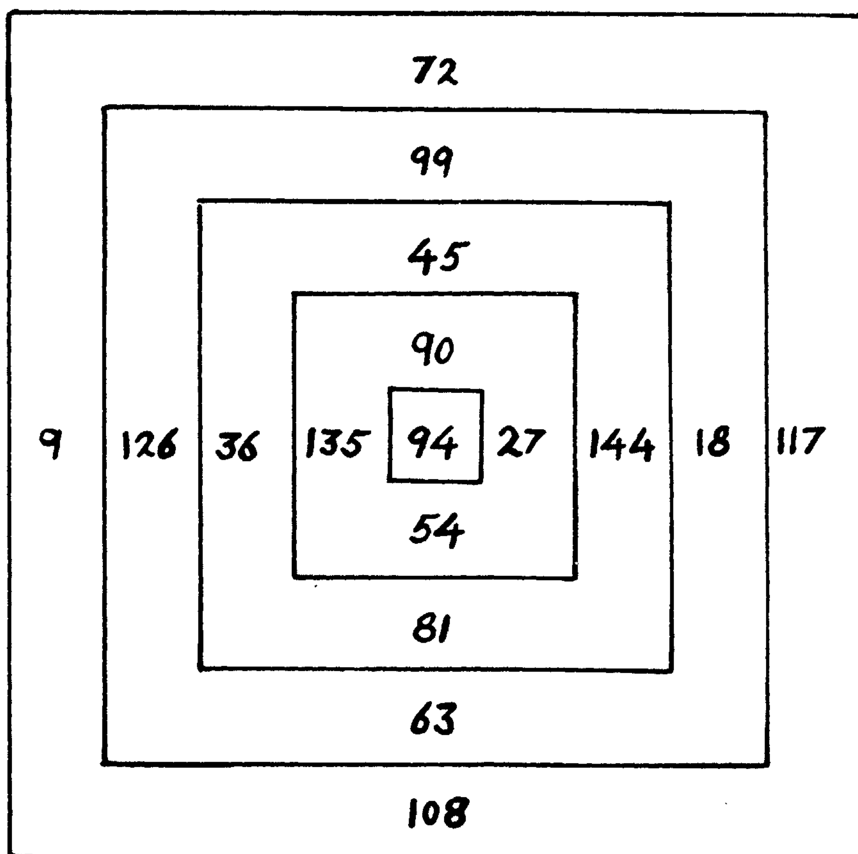
The square is obtained by multiplying each of the numbers of a 3 × 3 square using the natural numbers by 15. It will be further observed that (400 − 225) = 175 is placed in the centre, so that the sum of each of the arms radiating from the centre may be 400.

Magic Cross: The figure of the magic cross given by Nārāyaṇa is shown in Fig. 111.

This cross has been made with the help of the numbers of the 4 × 4 square given in Fig. 114. 94 has been placed in the centre to give the required total.

Magic Circles: Nārāyaṇa has given a number of magic circles each with total 400. These circles together with their key squares or rectangle are:

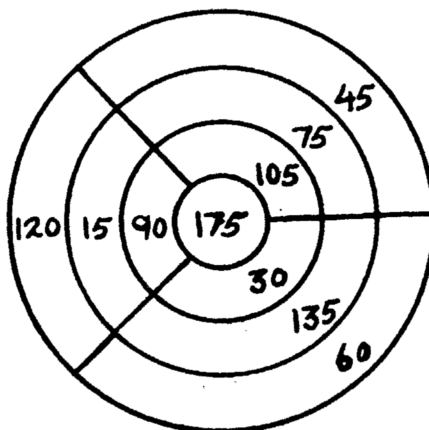
(i) *Magic Circle* from a 3 × 3 square using the series whose first term is 15 and common-difference 15 (Figs. 112 and 113).



Total = 400
Fig. 111

120	45	60
15	75	135
90	105	30

Total = 225
Fig. 112



Total = 400
Fig. 113

(ii) *Magic Circle* from a 4×4 square whose first term is 9 and common-difference 9 (Figs. 114 and 115).

9	72	117	108
126	99	18	63
36	45	144	81
135	90	27	54

Total = 306

Fig. 114

9
126
36
135
108 63 81 54 94 90 45 99 72
27
144
18
117

Total = 400

Fig. 115

(iii) *Magic Circle* from a 5×5 square whose first term is 4 and common-difference 4 (Figs. 116 and 117).

68	92	16	40	44
96	20	24	48	72
4	28	52	76	100
32	56	80	84	8
60	64	88	12	36

Total = 260

Fig. 116

16
24
52
80
88
40 48 76 84 12 140 64 56 28 20 92
36 60
8 32
100 4
72 96
44 68

Total = 400

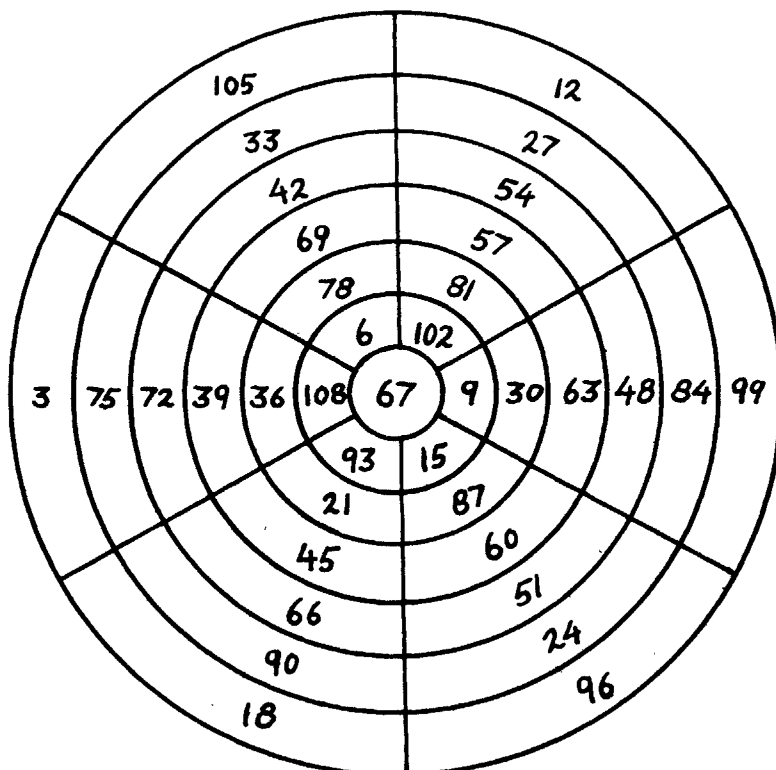
Fig. 117

(iv) *Magic Circle* from a 6×6 square whose first term is 3 and common-difference 3 (Figs. 118 and 119).

3	105	12	99	96	18
75	33	27	84	24	90
72	42	54	48	51	66
39	69	57	63	60	45
36	78	81	30	87	21
108	6	102	9	15	93

Total = 333

Fig. 118



Total = 400

Fig. 119

(v) *Magic Circle* from a 8×4 rectangle using the natural numbers 1 to 32 (Figs. 120 and 121).

1	16	25	24	2	15	26	23
28	21	4	13	27	22	3	14
8	9	32	17	7	10	31	18
29	20	5	12	30	19	6	11

Total rows 132, columns 66

Fig. 120

268

Total = 400

Fig. 121

Dharmanandana Square: Dharmanandana, a Jaina scholar (circa fifteenth century) has given[47] the following 8 x8 square[48] with total 260 (Fig. 122).

8	7	59	60	61	62	2	1
16	15	51	52	53	54	10	9
41	42	22	21	20	19	47	48
33	34	30	29	28	27	39	40
25	26	38	37	36	35	31	32
17	18	46	45	44	43	23	24
56	55	11	12	13	14	50	49
64	63	3	4	5	6	58	57

Fig. 122

The above square has been constructed by placing the natural numbers 1 to 64 in a 8×8 suqare in the direct order and then shifting the numbers so placed suitably. The square is divided into smaller squares of four cells each. The numbers in those squares that lie on the diagonals are unchanged, while those in the other squares are interchanged with the diagonally opposite ones. The manner of the change will be evident from the key square in Fig. 123 in which the smaller squares that are not to be interchanged are marked by thick letters and thick boundaries.

8	7	6	5	4	3	2	1
16	15	14	13	12	11	10	9
24	23	22	21	20	19	18	17
32	31	30	29	28	27	26	25
40	39	38	37	36	35	34	33
48	47	46	45	44	43	42	41
56	55	54	53	52	51	50	49
64	63	62	61	60	59	58	57

Fig. 123

Dharmanandana's method is quite general[49]. For instance, the 12 × 12 square shown in Fig.125 can be made by dividing the key square (Fig.124) into smaller squares of nine cells.

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99	100	101	102	103	104	105	106	107	108
109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132
133	134	135	136	137	138	139	140	141	142	143	144

Key-square
Fig. 124

The 4×4 magic square (Fig. 126) based on Dharmanandana's method is interesting as it is not included in Nārāyaṇa's squares (Fig. 36).

1	2	3	141	140	139	138	137	136	10	11	12
13	14	15	129	128	127	126	125	124	22	23	24
25	26	27	117	116	115	114	113	112	34	35	36
108	107	106	40	41	42	43	44	45	99	98	97
96	95	94	52	53	54	55	56	57	87	86	85
84	83	82	64	65	66	67	68	69	75	74	73
72	71	70	76	77	78	79	80	81	63	62	61
60	59	58	88	89	90	91	92	93	51	50	49
48	47	46	100	101	102	103	104	105	39	38	37
109	110	111	33	32	31	30	29	28	118	119	120
121	122	123	21	20	19	18	17	16	130	131	132
133	134	135	9	8	7	6	5	4	142	143	144

Total = 870

Fig. 125

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Key-square

Fig. 126(a)

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Total = 34

Fig. 126(b)

Sundarasūri Squares: Another Jaina scholar, Sundarasūri (circa fifteenth century), has given a number of interesting squares which have been constructed by novel methods[50]. An account of these squares is given below.

3×3 square: The filling of the 3×3 square as per Fig. 127 is according to the traditional Hindu method, already noted in Nārāyaṇa's work.

4×4 squares: A 4×4 square with any desired even total may be constructed by giving particular values to n in Fig. 128.

4	9	2
3	5	7
8	1	6

Total = 15
Fig. 127

n−8	n−1	2	7
6	3	n−4	n−5
n−2	n−7	8	1
4	5	n−6	n−3

Total = 2n
Fig. 128

Sundarasūri exhibits the instance with total 32 as per Fig. 129.

In Fig. 129, the number 8 occurs twice, because the total is less than 34, which is the least total for a 4×4 square constructed with the series of natural numbers.

Odd squares: Sundarasūri uses the elongated knight's move to obtain the 5×5 square shown in Fig. 130.

8	15	2	7
6	3	12	11
14	9	8	1
4	5	10	13

Total = 32
Fig. 129

22	3	9	15	16
14	20	21	2	8
1	7	13	19	25
18	24	5	6	12
10	11	17	23	4

Total = 65
Fig. 130

The method of filling is: Put 1 in the extreme cell of the middle row; move two cells in front and one cell diagonally, and put down the next number 2 and so on. When a block occurs, put the next number in the adjoining cell in the direction of the move, and continue as before[51].

The method can be easily generalised and is applicable to all odd squares. For filling up a $(2n+1) \times (2n+1)$ square the move to be used is n cells horizontally or vertically and one cell diagonally. When a block occurs, the next number is to be put down in front of the cell last filled in the direction of the move. By proceeding in this way, we obtain the required magic square. As an example, we give the 7×7 square as per Fig. 131.

20	28	29	37	45	4	12
44	3	11	19	27	35	36
26	34	42	43	2	10	18
1	9	17	25	33	41	49
32	40	48	7	9	16	24
14	15	23	31	39	47	6
38	46	5	13	21	22	30

Total = 175

Fig. 131

8×8 square: Sundarasūri gives the 8×8 square [52] shown in Fig. 132.

It has been constructed by dividing symmetrically the following key-square into groups of four and two cells. The numbers that lie in groups standing on the diagonals remain unchanged, while those in the others are interchanged with the diagonally opposite ones. The method of division will be apparent from Fig. 133 of the key-square:

1	63	62	4	5	59	58	8
56	10	11	53	52	14	15	49
48	18	19	45	44	22	23	41
25	39	38	28	29	35	34	32
33	31	30	36	37	27	26	40
24	42	43	21	20	46	47	17
16	50	51	13	12	54	55	9
57	7	6	60	61	3	2	64

Total = 260

Fig. 132

Compound magic squares: Sundarasūri gives the 9×9 square shown in Fig. 134.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Key-square

Fig. 133

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

Total = 369

Fig. 134

The method of construction of the above square is apparent from the figure if we consider the square to be divided into nine smaller squares, as is done in the figure given above. It will be found that each of the smaller squares is a 3×3 magic square. Therefore, the method is: Divide the numbers 1-81 into 9 groups in order, and with these groups construct nine 3×3 squares. These nine squares, being numbered one to nine in order, are filled in the bigger square just as in the method of filling a 3×3 square with the numbers 1-9[53].

Concluding Remarks

The foregoing pages would have shown to the reader that the Hindu achievements in the theory and construction of magic squares stand unsurpassed even up to the present day. The simplest square presenting any difficulty is the 4×4 square whose study began in India as early as the beginning of the Christian Era. The success obtained in constructing this square must have encouraged the consideration of larger squares. The construction of magic squares was not made a part of mathematics, as no theoretical treatment could be given in the earlier stages. There are, however, stray examples of the occurrence of magic squares from the beginning of the Christian Era right up to the time of Nārāyaṇa (1356)[54]. A very elegant and satisfactory method for the construction of the 4×4 square was

developed before the time of Nārāyaṇa. This method, which we may call the method of the knight's move, gives us 384 magic squares, which are perfect and possess the characteristics of what are now called "Nasik squares". This method of construction will be new to western scholars of today.

Nārāyaṇa (1356), who undertook the study of these squares, obtained results which have been only recently found in the west by the efforts of several workers. Of his theoretical results, the most important is the demonstration of the fact that magic squares may be constructed with as many series or groups of numbers in A.P. as there are cells in a column. This result was first stated in the west by L.S. Frierson in the beginning of the present century[55]. Another very important feature of Nārāyaṇa's work is the division of magic squares into three types. In our opinion, the recent work done in the west suffers from considerable inelegance because of the absence of such classification.

Nārāyaṇa claims as his own the methods for the construction of $4n \times 4n$ squares and odd squares by means of superposition, and also a method for the construction of $(4n+2) \times (4n+2)$ square. Methods for the construction of certain squares by means of superposition were devised by M de la Hir^ (1705)[56]. Nārāyaṇa's methods, given more than six centuries earlier, are more elegant and practical, although theoretically there is little difference between the two. Nārāyaṇa's method for the construction of $(4n + 2) \times (4n + 2)$ squares seems to be the only general method for the construction of such squares known up to the present.

The squares given by the Jaina monks Dharmanandana and Sundarasūri have evidently been obtained by generalisation of Nārāyaṇa's methods and show that the study of magic squares engaged the attention of the Hindus up to the fifteenth century.

The history of the development of magic squares in India, detailed in the preceding pages, leads irresistibly to the conclusion that the magic square originated in India. The knowledge of these squares might have gone outside India at any time between the first century and the tenth century AD. But it appears to be most probable that the west as well as China got the magic squares from India through the Arabs about the tenth century. This would account for the simultaneous occurrence of the magic square in such far off places as China, Arabia and Western Europe.

NOTES & REMARKS

1. The *Loh Shu* and the map of the Ho are illustrated in *Magic Squares and Cubes* by W.S. Andrews, Chicago, 1908, p. 122.
2. Cf. W.S. Andrews, l.c., p. 123.
3. In a work of Rabbi ben Ezra (c. 1140); cf. D.E. Smith, *History of Mathematics*, II, New York (1923), p. 596.
4. in the work of the Arab philosopher Gazzali, *cf.* Smith, D.E., l.c., p. 597.

5. It is said in the *Vedas* that the gods Indra and Viṣṇu divided 1000 into three. This incident is related in many works.(*Taittirīya Saṃhitā* vii. 1.6.; iii.2. 11; *Atharva-veda*, vii. 44.1.; *Taittirīya Brāhmaṇa*, i.1.6.1; *Śatapatha Brāhamaṇa* iii, 8.4.4. etc.). In the *Taittirīya Saṃhitā*, we have

"Ye twain have conquered; ye are not conquered,
Neither of the two of them hath been defeated;
Indra and Viṣṇu when contended,
Ye did divide the thousand into three."

(Keith)

"The thousand is divided into three at the three-night festival; verily he makes her possessed of a thousand, he makes her the measure of a thousand."

In the above passages it is not clear what "dividing a thousand into three" means. As the problem was considered so difficult that only the gods could solve it, so it is certain that "division into three" did not mean division into three equal parts or into any three parts or into three parts in arithmetic progression, for division as above can be easily made by the use of ordinary fractions which were known in those times. The passage very probably refers to the construction of a magic square with 1000 as total, especially as it has been stated that it confers benefits acting as a charm if the operation is performed at the three-night festival.

But to produce this passage as an evidence of the existence of magic squares, without other corroborative facts would, in our opinion, be as unjustifiable as the use of the *Loh-Shu* to establish the existence of the magic square in China in 2200 BC.

6. See *Indian Antiquary*, XI, 1882, pp. 83f.
7. Andrews. W.S., *Magic Squares and Cubes*, Chicago, 1908, p, 125f.
8. To fill this square the mnemonic formula stated by Nāgārjuna is:

Nīlaṃ^{30} cāpi^{16} dayā^{18}-calo^{36} naṭa^{10}-bhuvaṃ^{44} Khārī^{22}-varaṁ^{24} rāginam^{32}।

Bhūpo^{14} nāri^{20} vago^{34} jarā^{28}. cara^{26}-nibhaṃ^{40} tānaṃ^{06} śataṃ^{100} yojayet॥

9. *Bṛhat Saṃhitā* lxxvii. 23 ff.
10. The square as it actually occurs is interspersed with the *bīja* ("elements of a *mantra*").
11. Henceforth we shall translate this term by "number of the square".
12. *Gaṇita-Kaumudī* xiv 4. All the references that follow are from chapter xiv. To avoid unnecessary repetition, the number of the rules only, as found in Padmakara Dvivedi's edition of the *Gaṇita Kaumudī* will be given.
13. This seems to be the first statement of the result that magic squares are made from numbers in A.P., a result on which the whole theory of magic squares is founded.
14. Rule 5, the technical terms are: *mukha* = initial term and *pracaya* = common-difference.
15. Rule 6.
16. Rule 7.
17. Rule 8. The problem is indeterminate. The initial term is arbitrarily assumed, and the common-difference is obtained from the equation $s - n(n-1)d/2 = na$, where $s =$ the given sum, $n =$ the number of cells (or terms), $a =$ initial term, and $d =$ the common-difference.
18. *Caturbhadra* ("four magic square" or "4×4 magic square").
19. Rules 10-12. Nārāyaṇa ascribes the above method to previous writers. It cannot be said how old it is. The squares formed by the method are very popular among Hindu astrologers.
20. Rule 13 (a)
21. Rule 13 (b) − 14(a)
22. Ex. 4.
23. Rule 14(b)-15.
24. Examples 5. These are the first examples of squares constructed by a set of numbers not in a regular A.P.
25. Rules 16-20(a). If the number of *caraṇas* ("rows") be n, and if the first term be assumed to be a and the common difference d, the sum of n^2 terms divided by n, the number of rows, is the total

(*mukhaphala*) of the $n \times n$ square that will be constructed with this series. If the terms are written in rows of n, the initial terms of the rows, i.e., the *mukhapankti*, will be

a, (a + nd), (a + 2nd) ..., [a + n(n − 1) d]

Let the given total be T. The total corresponding to the *mukhapankti* (i.e.,*mukhaphala*) is

$$\frac{n^2\left\{a + \frac{(n^2-1)d}{2}\right\}}{n} = \frac{n}{2}\,[2a + (n-1)(n+1)\,d]$$

$$= \frac{n}{2}\left\{a + n(n-1)\,d] + a + (n-1)\,d\right\}$$

which is the form in which the total is expressed by Nārāyaṇa.

$$Kṣepaphala = T - \frac{n}{2}\;[a + n(n-1)\,d] + a + (n-1)d\}$$

$$= K, \text{(say)}.$$

We have now to find an arithmetic series of n terms whose sum is equal to K. The terms of this series are added to the corresponding terms of the *mukhapankti*. The rationale of the above result can be easily worked out. It can be easily seen that if A, D are the first term and the common-difference of the series whose sum is K, then the initial terms of the *caraṇas* ("*rows*") are

[a+A], [(a + nd) + A+D],....., [(a+n(n−1)d) + A+ (n−1)D]

26. Rules 20(b)-23(a).
27. Here, the term *gaccha* means the "number" of different sets of series that may be obtained for the filling of the square with the required total.
28. Nārāyaṇa gives these initial terms only.
29. Rule 23(b)-24(a).
30. The "number" of the square is the number of cells in a row of the square.
31. i.e., the upper horizontal half of the first square is filled first and then the lower half, and in the second square, the left vertical half is filled first.
32. Rules 24(b)-29. The author Nārāyaṇa was the son of Nṛsiṃha or Nṛhari.
33. It it convenient to take the *parāpankti* such that its sum is less than that of the *mūlapankti*, but this is not essential.
34. In (A'B'), the numbers are repeated. This is due to the fact that the *mūlapankti* and the *parapankti* in this case are the same.
35. This square is practically the same as Frost's "Nasik Square". (W.S. Andrews, l.c., p. 175, Fig. 288).
36. Rules 30-31.
37. It will be observed that all groups of 4 cells have the same total, except the groups included within the thick lines. If we interchange the third and fourth 4×4 squares, we get a 8×8 square in which all groups of 4 cells excepting the centre group have the total 130.
38. The *śliṣṭa* cells are cells not belonging to the diagonal and lying in the two vertical halves of the square. These cells are counted from the boundary inwards as will appear from the examples given. The number of such cells in a $(4n+2) \times (4n+2)$ square is n cells on the right and n cells on the left of each row.
39. Rules 32-36.
40. Rules 37-39.
41. Rules 41-42.
42. Rules 43-45.
43. See W.S. Andrews, l.c., p.1 ff.
44. Rules 46-49.
45. Rules 30-31.
46. i.e. any two lines of numbers that are side by side.

47. The square occurs in the *catuḥṣaṣṭi-yoginī-maṇḍala-stuti* of Dharmanandana.
48. This square is given in W.S. Andrews' book, l.c., Fig. 94, p. 43.
49. It is equally applicable to the smaller 4×4 square.
50. These squares occur in a *Stotra* by Sundarasūri.
51. W.S. Andrews gives the above method (p.4, Fig. 5) and claims it as his own. He has been anticipated by Sundarasūri by several centuries.
52. The same square has been given by W.S. Andrews, l.c., Fig. 53, p. 25. The square is perfect in all its characteristics. Sundarasūri's method can be generalised to obtain other squares.
53. The above method is now attributed to Prof. Hermann Schubert (cf.W.S. Andrews, l.c., Fig. 96, p. 44). In India it was known several centuries earlier.
54. Magic squares were used as charms and the method of construction seems to have been kept secret by the astrologers who used them in their trade. Another reason for their not occurring more frequently is that they did not belong to any particular subject and so had no place in the literature of the land.
55. Andrews, W.S., l.c., pp. 62 and 151-152.
56. *Memoires de l' Académie Royale* (1705). For a description of the method see also W.S. Andrews, l.c.

Indian Journal of History of Science, **27**(3), 1992.

USE OF PERMUTATIONS AND COMBINATIONS IN INDIA

BIBHUTIBHUSAN DATTA AND AWADHESH NARAYAN SINGH
(Revised by KRIPA SHANKAR SHUKLA)
Hussainganj Crossing, Behind Lakshman Bhawan, Lucknow-226001

(Received 24 December 1991)

Interest of the Hindus in the subject of permutations and combinations originated in connection with the variation of the Vedic metres in a very early age. There are specific rules for the calculation of the variation of metres in the *Chandaḥ-sûtra* of Piṅgala (before 200 BC). Permutations and combinations seem to have been subjects of such a fascinating study for the Hindus that they applied the ideas about them in various other spheres of life, e.g. architecture, music, medicine and astrology. Application of the principles of permutations and combinations is also found in the canonical literature of the Jainas in the study of philosophical categories. The present article aims at giving an account of the various uses of permutations and combinations in Indian literature.

EARLY INTEREST IN THE SUBJECT

The Hindu interest in the subject of permutations and combinations began in a very early age, first probably in connection with the variation of the Vedic metres and philosophical categories[1]. In the *Chandaḥ-sûtra* ("Rules of the Metre") of Piṅgala, a work on Vedic metres, written before 200 BC, we find specific rules for computation of the possible number of variations of even, semi-even, and uneven metres in a group with a specified number of long and short syllables in a quarter of a verse. In the *Nāṭya-śāstra* of Bharata Muni[2] are stated the number of variations of even metres having six to 26 syllables in a quarter of a verse. In the classical treatise on Hindu medicine by Suśruta, called *Suśruta-saṁhitā*[3], written about 600 BC, the total number of combinations that can be made out of six savours taking one, two, three,..., five and all at a time is found to have been correctly stated as 63. The early canonical literature of the Jainas (500-300 BC)[4] abounds in instances of speculation about the different sub-categories that can arise out of a fixed number of fundamental philosophical categories by the combinations of one, two, or more of them at a time. There are also similar calculations of the groups that can be formed out of the different instruments of senses, of the selections that can be made out of a number of males, females, and eunuchs or of permutations and combinations of various other things. The principles of the subject seem to have appealed to the Hindu mind and are found to have been applied in various spheres, such as astrology, perfumery, architecture, and music, besides those mentioned above. Thus, Bhāskara II (1150) observes: "It serves in prosody, for those versed therein, to find the variations of metres; in architecture to compute the changes in apertures, etc. (of a building); (in music), the scheme of musical permutations; and in medicine, the combination of different savours"[5].

Terminology

The oldest Hindu names for the subject of permutations and combinations are *vikalpa* (lit. "alternatives", "variations") and *bhaṅga* (lit. "poses"). Both these terms occur in the early canonical works of the Jainas (500-300 BC). The term *vikalpa* can be traced still earlier in the *Suśruta-saṁhitā* (*c.* 600 BC). Brahmagupta (628) calls it *Chandaściti* ("piling of metres"[6]), obviously because it originated, as has been stated above, in connection with the variation of Vedic metres. This name appears in later works also. Mahāvīra (850) calls combinations by the term *yutibheda* ("variations of combinations"[7]) and Śrīdhara (*c.* 750) and Bhaskara II (1150) by the term *bheda* or *vibheda* ("variation") only[8]. Bhāskara II introduces the names *aṅkapāśa* ("concatenation of numbers") and *gaṇita-pāśa*[9] for permutations. Nārāyaṇa (1356) has used the term *aṅka-pāśa*[10] to denote the whole subject of permutations and combinations. The Hindu expressions corresponding to the modern "taken one at a time", "taken two at a time", etc. are *ekaka-saṁyoga* (lit. "one-combination"), *dvika-saṁyoga* ("two-combination"), etc. These terms occur from the *Suśruta-saṁhitā* onwards. Other terms used in that sense are *eka-vikalpa* ("one variation"), *dvi-vikalpa* ("two variation"), etc.

Suśruta's Rules for Combinations

Suśruta (*c.* 600 BC)[11] states that the number of combinations of six savours — sweet, acid, saline, pungent, bitter and astringent — taken two at a time is 15. He seems to have arrived at it by writing down all the combinations exhaustively. For he observes: "On making two combinations in successive way, those beginning with sweet are found to be 5 in number; those beginning with acid are 4; those with saline 3; those with pungent 2; bitter and astringent make 1 combination". He then presents the actual combinations thus: sweet-acid, sweet-saline, sweet-pungent, sweet-bitter, sweet-astringent; acid-saline; acid-pungent; acid-bitter; acid-astringent; saline-pungent, saline-bitter, saline-astringent; pungent-bitter, pungent-astringent; and bitter-astringent. In the same way, Suśruta finds the number of 3-combinations to be 20; 4-combinations 15; 5-combinations 6; and 6-combinations 1. Thus, there are $6+15+20+15+6+1=63$ different combinations in all.

Jaina Canonical Works

In the early canonical works of the Jainas (500-300 BC), we find the results which correspond to

$$^{n}C_1 = n,\ ^{n}C_2 = \frac{n(n-1)}{1.2},\quad ^{n}C_3 = \frac{n(n-1)(n-2)}{1.2.3}, \ldots$$

After stating the results in case of $n=1, 2, 3, 4$, the *Bhagavatī-sūtra* observes: "And in this way 5, 6, 7, ..., 10, etc. numerable, innumerable, or infinite number of things may be mentioned. Forming one-combinations, two-combinations, three-combinations, and so on, ten-combinations, eleven-combinations, twelve-combinations, etc., as the successive combinations are formed, all of them should be considered[12]."

VARĀHAMIHIRA'S RULE

To find the number of combinations of n unlike things taken 1, 2, 3, ... at a time successively, Varāhamihira (*d.* 587) gives the following rule:

"They say that the number (of combinations) is obtained by (writing down the natural numbers 1, 2, 3, etc. up to the total number of things, one above the other, and) adding the preceding number to the succeeding one (in succession) and rejecting the last number [13]". The commentator Bhaṭṭotpala (966) has explained the process clearly by taking 16 different things. We reproduce from him the following scheme for it:

16			
15	120		
14	105	560	
13	91	455	1820
12	78	364	1365
11	66	286	1001
10	55	220	715
9	45	165	495
8	36	120	330
7	28	84	210
6	21	56	126
5	15	35	70
4	10	20	35
3	6	10	15
2	3	4	5
1	1	1	1

(The topmost number in the first column gives ${}^{16}C_1$, that in the second column gives ${}^{16}C_2$, that in the third column ${}^{16}C_3$, and that in the fourth column ${}^{16}C_4$. To get ${}^{16}C_5$ and others, the process of forming the successive columns should be continued further on.)

While dealing with the manufacture of perfumes in his *Bṛhat-saṁhitā*, Varāhamihira says:

"An immense number of perfumes can be made out of 16 ingredients, if every 4 of them are combined at will in one, two, three, and four proportions..... The total number of these perfumes will be 174720. Each substance (of a group of four) taken in one proportion being combined with the other three, taken in two, three, and four proportions, gives rise to 6 perfumes; and so it does, when taken in two, three or four proportions. One substance associated with a group of four substances (thus) gives rise to 24 perfumes; and in the same way the remaining three substances (of that group) (also give rise to 24 perfumes). The total of all these is 96. Now when 16 substances are divided into separate groups of 4 each, there arise 1820 such groups. Since each group of four gives rise to 96 varieties (of perfumes), therefore, that number (i.e., 1820) should be multiplied by 96. The number (resulting from this product) is the (total) number of perfumes[14]".

In another place, Varāhamihira states, "There are 31 varieties of *Anaphā-yoga* and *Sunaphā-yoga* each, and 180 of *Durudharā-yoga*[15]." Now it has been defined that an *Anaphā-yoga* occurs when one or more of the five planets, Mars, Mercury, Jupiter, Venus and Saturn, occupy the twelfth house from the Moon; in *Sunaphā-yoga*, a similar occurrence takes place in the second house from the Moon; and in the *Durudharā-yoga*, the planets occupy both these houses. Hence, we get

$$^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 5 + 10 + 10 + 5 + 1 = 31$$

$$\begin{aligned} &{}^5C_1({}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) + {}^5C_2({}^3C_1 + {}^3C_2 + {}^3C_3) + {}^5C_3({}^2C_1 + {}^2C_2) + {}^5C_4({}^1C_1) \\ &= 75 + 70 + 30 + 5 \\ &= 180. \end{aligned}$$

Brahmagupta (628) has devoted one full chapter (20th) of his treatise on astronomy, the *Brāhma-sphuṭa-siddhānta*, to the treatment of variation of metres. But on account of faulty readings, it has not been possible to make proper sense out of it.

Śrīdhara's Rule

To find the number of combinations of the six savours, taken one, two, three, ..., five, and all at a time, Śrīdhara gives the following rule:

"Writing down the numbers beginning with one and increasing by one up to the (given) numbers of savours, in the inverse order, divide them by the numbers beginning with one and increasing by one in the regular order, and then multiply successively by the preceding (quotient) the succeeding one[16]".

Thus, writing the numbers of the savours 1, 2, 3, 4, 5, 6 in the inverse order and dividing them by the same numbers in the regular order, we get

$$\frac{6}{1}, \quad \frac{5}{2}, \quad \frac{4}{3}, \quad \frac{3}{4}, \quad \frac{2}{5}, \quad \frac{1}{6}.$$

Performing the successive multiplication by the preceding quotient of the succeeding one, we get

$$\frac{6}{1}, \ \frac{6}{1}\times\frac{5}{2}, \ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}, \ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}, \ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}\times\frac{2}{5}, \ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}\times\frac{2}{5}\times\frac{1}{6}.$$

These are the values of $^6C_1$, $^6C_2$, $^6C_3$, ..., $^6C_6$ respectively.

Mahāvīra's Rule

To find the number of combinations of unlike things, Mahāvīra gives the following general rule:

"Set down the numbers beginning with unity and increasing by one, up to the (given) number (of things) in the regular and inverse orders in upper and lower rows respectively. The product of the numbers (in the upper row) taken right-to-left-wise being divided by the product of the (corresponding) numbers (in the lower row) taken in the same way, the quotient gives the result[17]".

That is to say, if there be n things, we shall have the arrangement

$$1, 2, 3, \ldots\ldots\ldots, n-r, n-r+1, \ldots\ldots, n-2, n-1, n$$

$$n, n-1, n-2, \ldots\ldots\ldots, r+1, r, \ldots\ldots, 3, 2, 1.$$

Then says Mahāvīra

$$^{n}C_{r} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3.\ldots\ldots\ldots r}$$

It is perhaps noteworthy that one of the illustrative examples given by both Śrīdhara and Mahāvīra is the same as that given by Suśrutā[18]. It appears also in Bhāskara II's *Līlāvatī*[19], and Nārāyaṇa's *Gaṇiṭa-kaumudī*[20].

Śrīśaṅkara's Rule

Bhaṭṭotpala (966) has quoted the following rule from another writer, probably Bhaṭṭa Śrīśaṅkara, of whom we know very little now:

"Write down (the natural numbers) in the reverse way and below them in the regular way. Multiply the numbers (in the two rows) taken in the regular way and divide the product from the upper row by that from the lower[21]".

So, the scheme in this case is

$$n, n-1, n-2, \ldots\ldots\ldots, n-r+1, n-r, \ldots\ldots, 3, 2, 1$$

$$1, 2, 3, \ldots\ldots\ldots, r, r+1, \ldots\ldots, n-2, n-1, n.$$

Bhāskara II's Rule

Bhāskara II (1150) says:

"Divide the numbers from one upwards, increasing by unity, set down in the inverse order, by the same (arithmeticals) written in the regular order. The first quotient, the second multiplied by the first, the next multiplied by that, and so on, give the combinations by one, two, three, etc. This is the general rule[22]".

An example from Bhāskara II:

"A pleasant, spacious and elegant palace, constructed by a skilful architect for the landlord, has eight apertures in it. Tell me the number of combinations of them formed by taking one, two, three, etc. (at a time)."

The total number of combinations

$= {}^8C_1 + {}^8C_2 + {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8$

$= 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1$

$= 255.$

Early Rule for Permutations

In the early canonical works of the Jainas, we find copious instances of calculation of permutations yielding results corresponding to the modern formulae.

${}^nP_1 = n$, ${}^nP_2 = n(n-1)$, ${}^nP_3 = n(n-1)(n-2)$, etc.

But the earliest mention of a rule for finding the number of permutations of n things taken all at a time is found in the *Anuyogadvāra-sūtra*, a canonical work written before the beginning of the Christian era. It says:

"What is the direct arrangement? *Dharmāstikāya, Adharmāstikāya, Ākāśāstikāya, Jīvāstikāya, Pudgalāstikāya* and *Addhāsamaya* — this is the direct arrangement. What is the reverse arrangement? *Addhāsamaya, Pudgalāstikāya, Jīvāstikāya, Ākāśastikāya, Adharmāstikāya* and *Dharmāstikāya* — this is the reverse arrangement. What are the mixed arrangements? Form the series of numbers beginning with one and increasing by one up to six terms. The mutual products of these minus 2 will give the number of mixed arrangements[24]".

We have similar rules for 7, 10, 16, 24 or any variable number (*asaṁkhyeya*) of unlike things[25]. Thus, it was known that the number of permutations of n unlike things taken all at a time is

$1.2.3\ldots(n-2)(n-1)n.$

Jinabhadra Gaṇi's Rule

Jinabhadra Gaṇi (529-589) says:

"Multiply mutually the numbers beginning with one and increasing by one up to the numbers of terms (i.e., unlike things); then the product (will give the number of permutations)[26]".

A similar rule has been given by the commentator Śīlāṅka (862) from an unknown writer:

"Beginning with unity up to the number of terms, multiply continuously the (natural) numbers. The product should be known as the result (i.e. the total number) in the calculation of permutations (*vikalpagaṇita*)[27]".

Bhāskara II's Rules

To find the number of permutations of n unlike things taken all at a time, Bhāskara II (1150) gives a rule similar to those stated above:

"The product of the numbers beginning with and increasing by unity and continued up to the number of places will be the number of different permutations with all of the specified things[28]".

He then gives a rule for finding the permutations of n unlike things taking any variable number of them at a time.

"The product of the numbers from the total number of places and decreasing by unity, continued up to the last of the (variable) places gives the number of permutations of unlike things[29]".

That is to say, the number of r permutations of n dissimilar things will be

$$n(n-1)(n-2).....\text{up to r factors.}$$

Similar rules are given by Nārāyaṇa[30].

Permutations of Things Not All Different

To find the number of ways in which n things may be arranged amongst themselves, taking all at a time, when some of the things are alike, Bhāskara II gives the following rule:

"Find separately the number of permutations for as many places as are occupied by like digits; then divide by that the number of permutations calculated before (on the supposition that all the digits are unlike): the quotient will be the (required) number of permutations[31]".

A similar rule is given by Nārāyaṇa[32].

"That is to say, if p of the digits are alike of one kind, q of them are alike of a second kind, r of them are alike of a third kind, and the rest all different, then the number of

permutations will be

$$\frac{n!}{p!\,q!\,r!},$$

n being the total number of places occupied by the digits (like and unlike).

Examples from Bhāskara II[33]

The different numbers that can be formed out of the digits 2, 2, 1, 1 are in all

$$\frac{4!}{2!\,2!}=6.$$

The various numbers that can be formed out of the digits 4, 8, 5, 5, 5 are altogether

$$\frac{5!}{3!}=20.$$

Nārāyaṇa[34] states that when each of the n things is repeated, the number of r-permutations is n^r. As examples, he finds that with the digits 1 and 2 there can be formed as many as 2^6 or 64 numbers of six notational places each, and with the digits 1, 2 and 3 will be obtained 3^3 or 27 numbers of three notational places each[35].

SUM OF PERMUTATIONS

To find the sum of the numbers that can be formed by the permutations of some given digits, taken all at a time, Bhāskara II gives the following rule:

"That (the number of permutations) is divided by the number of digits and multiplied by their sum; the result being repeated according to the notational places (as many times as the number of digits) and added together will give the sum of the permuted numbers[36]".

This rule is equally applicable to both the cases when all the digits are unlike and when some of them are alike[37].

Illustrative examples from Bhāskara II[38]

(1) The numbers that can be formed by permutation of the eight digits 2, 3, 4, 5, 6, 7, 8, 9 are altogether

$$=1.2.3.4.5.6.7.8.$$

$$=40320$$

Now we have

$$\frac{40320}{8}(2+3+4+5+6+7+8+9)=221760;$$

also setting down 221760 eight times advanced forward one place each time and then adding together, we get

```
221760
 221760
  221760
   221760
    221760
     221760
      221760
       221760
-------------
2463999975360
```

Hence, the sum of the numbers obtained by permutation is 2463999975360.

(2) The number that can be formed by the digits 2,2,1,1 has been found to be equal to 6 altogether. Now, we get

$$\frac{6}{4}(2+2+1+1)=9;$$

and also

```
9
 9
  9
   9
----
9999
```

Hence, the required sum in 9999.

The *rationale* of the rule is as follows[39].

Case 1. Suppose there are n digits and all of them are *unlike*.

The number of permutations that can be formed with these digits is n!. Now consider any of the digits, say a. In (n − 1)! of the numbers a will be in the units' place; in as many cases it will be in the tens' place; and so on. The sum arising from a alone, since there are n digits in all,

$$= (n-1)!(10^{n-1}a + 10^{n-2}a + ... + 10a + a)$$

$$= (n!/n)\ (10^{n-1} + 10^{n-2} + ... + 10 + 1)a$$

Proceeding in the same way with the other digits and adding up the partial sums, we get the sum of all the numbers resulting from permutations of the digits

$$= (n!/n)\ (10^{n-1} + 10^{n-2} + ... + 10 + 1)\ (\text{sum of the digits})$$

Case 2. Suppose p of the digits to be alike and equal to k_1, q of them equal to k_2, r of them equal to k_3 and the rest unlike.

The number of permutations that can be made with these digits is

$$\frac{n!}{p!q!r!}$$

The number of cases in which k_1 is in the units' place is

$$\frac{(n-1)!}{(p-1)!q!r!}$$

In as many cases it is in the tens' place; and so on. Hence, the partial sum arising out of k_1 is

$$\frac{(n-1)!}{(p-1)!q!r!}\ (10^{n-1} + 10^{n-2} + ... + 10 + 1)k_1$$

In the same way, the partial sums arising from k_2 and k_3 are respectively

$$\frac{(n-1)!}{(q-1)!p!r!}\ (10^{n-1} + 10^{n-2} + ... + 10 + 1)k_2$$

$$\frac{(n-1)!}{(r-1)!p!q!}\ (10^{n-1} + 10^{n-2} + ... + 10 + 1)k_3$$

and the partial sum due to the unlike digits k_4, k_5,.....is, by Case 1

$$\frac{(n-1)!}{p!q!r!}\ (10^{n-1} + 10^{n-2} + ... + 10 + 1)\ (k_4 + k_5 + ...)$$

Hence, the required sum of all the numbers is

$$\frac{n!}{n\ p!q!r!}\ (10^{n-1} + 10^{n-2} + ... + 10 + 1)\ (pk_1 + qk_2 + rk_3 + k_4 + k_5 + ...)$$

$$= \frac{n!}{n\,p!q!r!}(10^{n-1}+10^{n-2}+\ldots\ldots+10+1) \text{ (sum of all the digits).}$$

Bhāskara II's Problem

Bhāskara II proposed an interesting problem: To find how many different numbers occupying a specified number of notational places can be formed out of digits having a definite sum. His solution is as follows:

"When the sum of the digits is fixed, divide the successive numbers beginning with that sum minus one, and decreasing by one, continued up to one less than the number of places, by one, two, etc. respectively. The variations of numbers will be equal to the product of those quotients. This rule is valid, it must be known, only when the sum of the digits is less than the specified number of notational places plus nine".[40]

Illustrative example from Bhāskara II[41]

The different numbers of 5 digits of sum 13 will be altogether

$$\frac{12}{1}\cdot\frac{11}{2}\cdot\frac{10}{3}\cdot\frac{9}{4} = 495.$$

Representation

It has been noted before that the interest of the ancient Hindus in the subject of permutations and combinations was not of theoretical origin, but grew out of a concrete purpose. For that it was essential not only to know the number of possible variations but also, and in a greater degree, to have the actual variations. So, we find that as early as the time of the Jaina canonical works, distinct consideration was being made between *bhaṅga-samutkīrṇatā* ("Telling permutations or combinations", that is, "Enumeration of possible variations") and *bhaṅga-pradarśanatā* ("Representation of permutations and combinations"). In the early state of the subject even the number of variations in any given case very probably used to be determined by writing them all down exhaustively. But the latter was obviously a laborious task and was often liable to be in error if all the operations be not carried out in a systematic way. Such a systematic scheme of operations is technically called the *Loṣṭa-prastāra* ("Spreading out of marked objects"), apparently because in the beginning the permutations or combinations used to be formed out of any given number of things by laying out objects, probably clay pieces, marked with the tachygraphic abbreviations of the names of the various things.

Representation of Combinations

A scheme of writing down all the possible combinations formed out of a given number of unlike things is sufficiently clear from the descriptions of Suśruta. The same appears in the Jaina canonical works[42]. Varāhamihira's rule for that is as follows:

"Any one of the things taken optionally should be successively operated upon (by the rest); when that process is exhausted, the next (should be begun)"[43].

The operations implied have been explained at length by Bhaṭṭotpala with the help of specific instances. In this connection he has quoted a rule from Bhaṭṭa Śrīśaṅkara[44]. Jinabhadra Gaṇi (*c.* 550) also has a rule for the same[45].

Representation of Permutations

Śīlāṅka (862)[46] has quoted a rule from an ancient writer who is not known now, describing a systematic scheme of forming all the possible permutations out of a given number of unlike things:

"The total number of permutations should be divided by the last term, then the quotient by the rest. They should be placed successively by the side of the initial term in the calculation of permutations."

The rule appears to be cryptic, but Śīlāṅka has explained it clearly with the help of an illustrative example: To find the numbers that can be formed by using the digits 1, 2, 3, 4, 5, 6. It is as follows:

Let there be n number of things a_1, a_2, ..., a_n. Then the total number of permutations that can be formed out of them will be n!. The number of permutations which can have any particular thing, say a_1, for its initial digit (*ādi*) will be n!/n, that is, (n − 1)!. So, put a_1 in the beginning of (n − 1)! grooves and so on. Again amongst the first series of grooves, the number of sub-grooves that can have a_2 after a_1 will be (n − 1)!/(n − 1) or (n − 2)!. Place a_2 after a_1 in those sub-grooves. The number of sub-grooves that can have a_3 after a_1 will be (n − 2)! and put it after a_1 in those sub-grooves. Similarly, with a_4, a_5,...., a_n. Again amongst the sub-grooves that can have any other particular thing in the third place will be (n − 3)! and it should be placed in those cases. Proceeding step by step in this way in a systematic manner, we can find out all the different permutations of things.

Piṅgala's Rules

Piṅgala (before 200 BC) describes a scheme of forming all the permutations with a specified number of things when repetitions are allowed. As he was directly concerned with metres, he dealt with only two varieties of things, long and short syllables, which are represented respectively by the abbreviations *g* from *guru* ("long") and *l* from *laghu* ("short"). But the scheme is equally applicable to cases of more varieties. Piṅgala's scheme, described in short aphorisms[47], will be clear from the following:

(i) Monosyllabic: 1. *g*
2. *l*

(ii) Disyllabic:

$$\left.\begin{array}{l}\left.\begin{array}{l} g \\ l \end{array}\right\} g \\ \left.\begin{array}{l} g \\ l \end{array}\right\} l \end{array}\right\} = \left\{\begin{array}{ll} 1. & gg \\ 2. & lg \\ 3. & gl \\ 4. & ll \end{array}\right.$$

(iii) Trisyllabic:

$$\left.\begin{array}{l}\left.\begin{array}{l} gg \\ lg \\ gl \\ ll \end{array}\right\} g \\ \left.\begin{array}{l} gg \\ lg \\ gl \\ ll \end{array}\right\} l \end{array}\right\} = \left\{\begin{array}{ll} 1. & ggg \\ 2. & lgg \\ 3. & glg \\ 4. & llg \\ 5. & ggl \\ 6. & lgl \\ 7. & gll \\ 8. & lll \end{array}\right.$$

and so on. Piṅgala states that the trisyllabics are 8 in number[48]. In general, a group of n syllables will have 2^n forms (*vide infra*).

The above systematic scheme of representation has the advantages that (*a*) we can easily find out the form of versification corresponding to a given serial number in it and vice versa, (*b*) we can allocate a given form of versification in its proper place in the scheme. Piṅgala's aphorisms for (*a*) are, "*l* when halved; *g* when added with one (and then halved)"[49]. That is to say: Divide the given number successively by two; if at any step, the number obtained is not divisible by two, add one to it and then halve. Corresponding to each operation of exact division by two, set down *l*; and to that of halving after adding unity write down *g*. The operations are to be continued until the desired number of syllables in the group has been obtained. The operations for (*b*) are the reverse of these [50]. Taking unity, we shall have to double it successively as many times as there are syllables in the given form; but corresponding to each long syllable we shall have to subtract one from the corresponding product.

Piṅgala next gives a rule for finding the total number of variations without having recourse to writing them all down exhaustively according to the scheme described above. This method has already been described. It is found in later writings also[51]. By this rule, the total number of variations in a group of n syllables is found to be equal to 2^n.

Piṅgala has also an alternative method to find the total number of variations[52]. It is technically called *Meru-prastāra*, because the total is obtained by addition from numbers arranged in such a form as to present a fancied resemblance to the fabulous mountain Meru of the Hindu mythology. Piṅgala's aphorisms being too compressed and cryptic can be understood only with the help of a commentary. Halāyudha (10th century) has explained them as follows:

"Draw one square at the top; below it draw two squares, so that half of each of them lies beyond the former on either side of it. Below them in the same way draw three squares; then below them four; and so on up to as many rows as desired: this is the preliminary representation of the Meru (*Meru-prastāra*). Then putting down one in the first square, the marking should be started. In the next two squares write one in each. In the third row, put 1 in each of the two extreme squares and in the middle square, the sum of the two digits in the two squares of the second row. In the fourth row, put 1 in the two extreme squares; in an intermediate square put the sum of the digits in two squares of the previous row, which lie just above it. Putting down numbers in the other rows should be carried on in the same way. Now the numbers in the second row of squares show the monosyllabic forms: There are two forms, one consisting of a long and the other of a short syllable. The numbers in the third row give the disyllabic forms: in one form all syllables are long; in two forms one syllable is short; and in one all syllables are short. In this row of the squares we get the number of variations of the even verse. The numbers in the fourth row of squares represent trisyllabics. There one form has all syllables long, three have one short syllable; three have two short syllables and one has all syllables short, and so on. In the fifth and succeeding rows also the figure in the first square gives the number of forms with all syllables long, that in the last all syllables short and the figures in the successive intermediate squares represent the number of forms with one, two, etc. short syllables"[53].

Thus, according to the above, the number of variations of a metre containing n syllables will be obtained from the representation of the *Meru* as follows:

Number of syllables		Total number of variations	
	1		
1	1 1	2	2^1
2	1 2 1	4	2^2
3	1 3 3 1	8	2^3
4	1 4 6 4 1	16	2^4
5	1 5 10 10 5 1	32	2^5
6	1 6 15 20 15 6 1	64	2^6

From the above it is clear that Piṅgala knew the results:

(1) ${}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1} + {}^nC_{n+1} = 2^n$,

(2) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

Sanskrit prosody distinguishes three classes of metres: (1) even, in which the arrangement of syllables in all the quarters (*padas*) is the same; (2) semi-even, in which the alternate quarters are alike; and (3) uneven, in which the quarters are all dissimilar. Now with a group of n syllables in a quarter, the total number of varieties of even metres

will be, according to Piṅgala[54], 2^n; semi-even, $2^{2n}-2^n$; and uneven, $2^{4n}-2^{2n}$. The same formulae are stated also by Bhāskara II:

"The number of syllables in a quarter being taken for the period and the common ratio 2 the result from multiplication and squaring[55] will give the number of even metres. Its square, and square's square, minus their respective roots, will be the numbers of semi-even and uneven metres respectively"[56].

By way of illustration, Halāyudha[57] calculates that in the *Gāyatrī* metre, which has six syllables in a quarter, the number of even variations will be 64, semi-even 4032, and uneven 16773120. Bhāskara II[58] calculates that in the case of the *Anuṣṭubh* metre, which has 8 syllables in a quarter even variations are 256, semi-even 65280, and uneven 4294901760.

NEMICANDRA'S RULES

We find in the works of Nemicandra, a Jaina philosophical writer of the tenth century (*c.* 975), certain interesting rules, some of which are akin to those of Piṅgala. According to the Jaina philosophy, there are 15 kinds of *pramāda* ("carelessness"), of which four belong to the category of *vikathā* ("wrong talk"), four to that of *kaṣāya* ("passion"), five to that of *indriya* ("sense") and one each to those of *nidrā* ("sleep") and *praṇaya* ("attachment"). Combinations are made of five elements of carelessness, selecting only one element from each of the five categories. Again, they are formed by setting down the elements according to a systematic scheme and are marked serially. Hence, the problems that arise in this connection are, as enumerated by Nemicandra, to find: (1) the total number of combinations that can be made, (ii) a systematic scheme of laying out, (iii) the elements of a combination from its serial number, and (iv) the serial number of a particular combination[59]. Nemicandra has given rules for each.

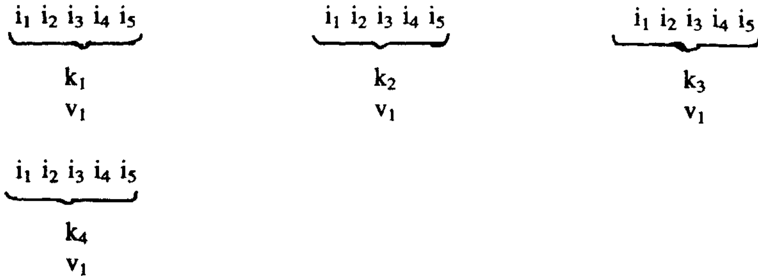
(i) "All the combinations previously obtained combine with each element of the next category. Hence, the total number will be given by the multiplication (of the numbers of elements in the different categories)"[60].

Thus, the total number of combinations that can be made out of the 15 elements of carelessness in the way described above is $4 \times 4 \times 5 \times 1 \times 1 = 80$.

(ii) Nemicandra has described two schemes of representation of combinations: one is called (1) the *prastāra* and the other (2) the *parivartana*.

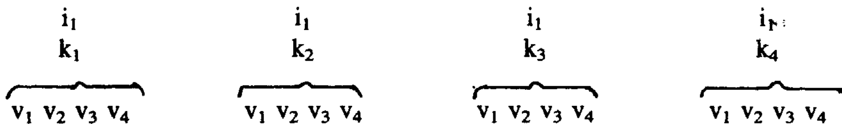
(1) "The distribution will be obtained thus: Write down severally the first element (of the first category) of carelessness and put over it each of the elements of the succeeding classes. When elements in the third category are exhausted, begin afresh with the second element of the second category (and so on). When all the elements of these two categories are thus distributed out, operations should be begun with (the second element of) the first category. (And so on)"[61].

If the four kinds of wrong talks be denoted by v_1, v_2, v_3, v_4; the four kinds of passions by k_1, k_2, k_3, k_4; and the five kinds of senses by i_1, i_2, i_3, i_4, i_5, the representation described here will be this:



and so on, v_2, v_3, v_4 coming successively in the places of v_1.

(2) "Write down the elements of the first category as many times as the number of elements in the second category; then put down over each group severally each of the elements of the second category; and proceed thus throughout. When all the elements of the first category are exhausted, begin afresh with the second category (and so on). When all the elements of these two categories are thus distributed out, the operation with the elements in the third category begins"[62].



(iii) "Divide (the given serial number) successively by the number of elements in the different categories, adding each time unity to the quotient, except when the remainder is zero. The remainders determine the place of an element in its category; the zero remainder indicates the last element"[63].

For example, let us find the elements of the 13th combination.

First Scheme: Dividing 13 by 5, we get 2 for the quotient and 3 as the remainder. So, the combination contains the element i_3. Adding 1 to the quotient 2, we have 3. Dividing 3 by 4, we get the quotient 0 and the remainder 3. Hence, there is the element k_3, $0+1=1$. On dividing 1 by 4, the remainder is 1. So, there is v_1. So, the 13th combination according to the first scheme contains i_3, k_3 and v_1, besides sleep and attachment.

Second scheme: On dividing 13 by 4, the quotient is 3 and the remainder 1. So, the combination contains v_1. Adding unity to the quotient 3, we get 4. On dividing 4 by 4, the quotient is 1 and the remainder 0. Hence, there is k_4[64]. Dividing 1 by 5, we get the remainder 1. So, there is i_1. Hence, the 13th combination according to the second scheme has v_1, k_4, i_1, besides sleep and attachment.

(iv) "Take unity. Multiply it by the total number of elements in a category beginning from the last and subtract from the product the number of elements there following the given element. Proceed in the same way throughout"[65].

For example, let us find the number of the combination $i_4k_3v_1$. Take 1. As there are 4 elements in the last category v, we multiply it by 4 and get $1 \times 4 = 4$. Since there are only 3 elements in that category after v_1, we subtract 3 from the product and get $4 - 3 = 1$. Next, we shall have to multiply the remainder 1 by 4, since there are 4 elements in the category of k and subtract from the result 1, since there lies only 1 element in the category after k_3. Thus, we get $1 \times 4 - 1 = 3$. Now we multiply 3 by 5, there being 5 elements in the category of i and then subtract 1, there being only one element after i_4. So, we get $3 \times 5 - 1 = 14$. Hence, the serial number of the combination $i_4k_3v_1$ is 14.

To get the same results as stipulated in rules (iii) and (iv) more easily and quickly, without going through the lengthy process of calculations described therein, Nemicandra gives two short tables. He says:

Table 1

"Place 1, 2, 3, 4, 5; 0, 5, 10, 15; 0, 20, 40 and 60 in three rows (of cells) of the three categories of carelessness, and find the elements and the serial numbers of combinations"[66].

i_1 1	i_2 2	i_3 3	i_4 4	i_5 5
k_1 0	k_2 5	k_3 10	k_4 15	
v_1 0	v_2 20	v_3 40	v_4 60	

Table 2

"Set down 1, 2, 3, 4; 0, 4, 8, 12; 0, 16, 32, 48 and 64 in three rows (of cells) of the three categories of carelessness, and find the elements and the serial numbers of combinations"[67].

v_1 1	v_2 2	v_3 3	v_4 4	
k_1 0	k_2 4	k_3 8	k_4 12	
i_1 0	i_2 16	i_3 32	i_4 48	i_5 64

Table 1 is to be used in case of distribution on the first scheme and Table 2 in that on the second scheme. To find the serial number of a given combination, we have simply to add together the figures placed in the cells of its elements in the tables. And to determine the elements occurring in a combination whose serial number is given, we shall have to break up that number into three parts picked up from three rows of cells in the tables and then write down in order the elements from those cells.

For example, since $13 = 3 + 10 + 0$, the 13th combination in the first scheme will be $i_3k_3v_1$ as determined before. According to the second scheme, it will be $v_1k_4i_1$, since $13 = 1 + 12 + 0$.

Notes and References

1. See the article of Gurugovinda Chakravarti on the "Growth and development of permutations and combinations in India" in *BCMS*, XXIV (1932).
2. See Ch. xiv, vv. 55-81.
3. See Ch. lxiii.
4. For instance see *Jambūdvīpa-prajñapti* xx. 4, 5; *Bhagavatī-sūtra, Sūtras* 314, 341, 371-4, etc.; *Anuyogadvāra-sūtra, Sūtras* 76, 92, 126. Compare Bibhutibhusan Datta, "The Jaina School of Mathematics", *BCMS,* XXI (1929), pp. 133 ff.
5. See *L* (= *Līlāvatī*) p. 26 f. *Cf. GK* (= *Gaṇita-Kaumudī*), xiii. 2.
6. See *BrSpSi* (= *Brāhma-sphuṭa-siddhānta*), xx.
7. See *GSS* (= *Gaṇita-sāra-saṅgraha*), Ch. vi.
8. See *PG* (= *Pāṭīganita*), p. 95.
9. See *L*, p. 83.
10. See *GK*, II, p. 286.
11. In *Suśruta-saṁhitā*, Ch. lxiii.
12. *Bhagavatī-sūtra, Sūtra* 314.
13. *Bṛhat-saṁhitā*, with the commentary of Bhaṭṭotpala, edited by Sudhākara Dvivedī, in two volumes, Benaras, 1897, lxxvi, 22.
14. *Bṛhat-saṁhitā*, lxxvi. 13-21. See also lxxvi. 29-30.

    It will be noted that the total number of perfumes will be 24×1820, i.e. 43680, and not 174720, as stated by Varāhamihira. His commentator Bhaṭṭotpala rightly remarks: "This number (i.e. 174720) is obtained by taking all varieties subordinate to each ingredient, and not by taking the main varieties (which must be all different). Considering the main varieties only, the total number of perfumes comes to 43680, because a group of four yields only 24 varieties (of perfumes)".
15. *Bṛhajjātaka*, edited by Sitarama Jha, with the commentary of Bhaṭṭotpala, Benaras, 1921, Ch. xiii, vs. 4.
16. *PG*, Rule 72.
17. *GSS*, vi. 218.
18. PG, Ex. 95, *GSS*, vi. 19.
19. *L*, p. 27.
20. *GK*, xiii. Ex. 22.
21. *Bṛhajjātaka*, xii. 19(com).
22. *L*, p. 27.
23. *GK*, xiii. 59.
24. *Anuyogadvāra-sūtra, Sūtra* 97.
25. *Ibid, Sūtras* 103, 114-9.
26. *Viśeṣāvaśyaka-bhāṣya, Gāthā* 942.
27. *Vide* Śīlāṅka's comm.on *Sūtrakṛtāṅga sūtra, samayādhyayana, anuyògadvāra*, verse 28.
28. *L*, p. 83.
29. *L*, p. 84.
30. *GK*, xiii. 45, 91.

31. *L*, p. 84.
32. *GK*, xiii. 55(c – d)-56(a – b).
33. *L*, p. 84.
34. *GK*, xiii. 62(a).
35. *GK*, xiii. Ex. 27.
36. *L*, p.83.
37. *L*, p. 84.
38. *L*, pp. 83, 84.
39. *Cf.* Haran Chandra Banerjee; *Līlāvatī*, Second edition, Calcutta (1927), pp. 192-195.
40. *L*, p.85.
41. *L*, p. 85.
42. *Bhagavatī-sūtra, Sūtra* 314.
43. *Bṛhat-saṁhitā*, lxxvi. 22; *Bṛhajjātaka*, xiii. 4.
44. Vide Bhaṭṭotpala's commentary on *Bṛhajjātaka*, xii. 19.
45. *Viśeṣāvaśyaka-bhāṣya*.
46. loc. cit; *Cf.* B. Datta, *Jaina Math*, pp. 135 f.
47. *Chandaḥ-sūtra of Piṅgala*, edited by Jīvananda Vidyasagara, with the commentary of Halāyudha, Calcutta, 1892; viii. 20-2.
48. *Chandaḥ-sūtra of Piṅgala*, viii. 23.
49. *Ibid*, viii. 24-5.
50. *Ibid*, viii. 26-7.
51. For instance, Mahāvīra (*GSS*, ii. 94). Pṛthūdakasvāmī (BrSpSi, xii. 17. com.), etc.
52. *Chandaḥ-sūtra*, viii. 28-32.
53. *Chandaḥ-sūtra*, viii. 33-4.
54. *Chandaḥ-sūtra*, v. 3-5.
55. Reference here is to the operations described for finding the sum of a G.P. (*L*, p. 31).
56. *L*, p. 31.
57. See his commentary on *Chandaḥ-sūtra*, v. 3-5.
58. *L*, p. 32.
59. *Gommaṭasāra, Jīvakāṇḍa, Gāthā* 35.
60. *Ibid, Gāthā* 36.
61. *Gommaṭasāra, Jīvakāṇḍa, Gāthās* 37, 39.
62. *Ibid, Gāthās* 38, 40.
63. *Ibid, Gāthā* 41.
64. As there is no element with zero suffix, the remainder gives k_4.
65. *Ibid, Gāthā* 42.
66. *Ibid, Gāthā* 43.
67. *Ibid, Gāthā* 44.

Indian Journal of History of Science, 28(2), 1993

USE OF SERIES IN INDIA

BIBHUTIBHUSAN DATTA and AWADHESH NARAYAN SINGH
(Revised by KRIPA SHANKAR SHUKLA)
Argara, **Hussainganj** Crossing, Behind Bata Shoe Co.
Lucknow-226 001

(Received 24 December 1991)

Particular instances of arithmetic and geometric series have been found to occur in Vedic literature as early as 2000 BC. From Jaina literature it appears that the Hindus were in possession of the formulae for the sum of the arithmetic and the geometric series as early as the fourth century BC, or earlier. In the *BakhshaliManuscript* and other works on *Pāṭīgaṇita*, series were treated as one of the major topics of study and a separate section was generally devoted to the rules and problems relating to series. In Europe, the series were looked upon as one of the fundamental operations, evidently due to Hindu influence through the Arabs. Besides the arithmetic and the geometric series, a number of other types of series, e.g., the series of sums, the series of squares or cubes of the natural numbers, the arithmetico-geometric series, the series of polygonal or figurate numbers, etc. occur in the works on *Pāṭīgaṇita*. There is, however, no mention of the harmonic series.

Evidence of the use of the infinite geometric series with common ratio less than unity is found in the ninth century. The formula for the sum of this series was known to the Jainas who used it to find the volume of the frustum of a cone. The Kerala mathematicians of the fifteenth century gave the expansions of sin x, cos x, tan x, and π long before they were known in Europe or anywhere else.

The present article gives an account of the use of series in Indian literature.

ORIGIN AND EARLY HISTORY

Series of numbers developing according to certain laws have attracted the attention of people in all times and climes. The Egyptians are known to have used the arithmetic series about 1550 BC[1]. Arithmetic as well as geometric series are found in the Vedic literature of the Hindus (*c.* 2000 BC). In the *Taittirīya-Saṃhitā*[2] we find the series:

(i) 1, 3, 5, ..., 19, 29, ..., 99

(ii) 2, 4, 6, ..., 20

(iii) 4, 8, 12, ...

(iv) 10, 20, 30, ...

(v) 1, 3, 5, ..., 33

In the *Vājasaneyī Saṃhitā*[3], we have the *yugma* ("even") and *ayugma* ("odd") series:

(vi) 4, 8, 12, 16, ..., 48

(vii) 1, 3, 5, 7, ..., 31.

The *Pañcaviṃśa Brāhmaṇā*[4] has the following geometric series:

(viii) 12, 24, 48, 96, ..., 196608, 393216.

Another geometric series occurs in the *Dīgha Nikāya*[5]. It is

(ix) 10, 20, 40, ..., 80000.

The Hindus must have obtained the formula for the sum of an arithmetic series at a very early date, but when exactly they did so cannot be said with certainty. It is, however, definite that in the 5th century BC, they were in possession of the formula for the sum of the series of natural numbers, for in the *Bṛhaddevatā* (500-400 BC)[6] we have the result

$$2 + 3 + 4 + \ldots + 1000 = 500499.$$

In the *Kalpa-sūtra* of Bhadrabāhu (*c.* 350 BC), we have the sum of the following geometric series

$$1 + 2 + 4 + \ldots + 8192 \text{ (i.e., to 14 terms)}$$

given correctly as 16383, showing that the Hindus possessed some method of finding the sum of the geometric series in the 4th century BC.

The following result occurs in the commentary, entitled, ***Dhavalā***[7], by Vīrasena (*c.* 9th century AD) on the Ṣaṭkhaṇḍāgama of Puṣpadanta Bhūtabali:

$$49\frac{217}{452}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots \textit{ ad inf.}\right) = 65\frac{110}{113}.$$

This shows that the following formula giving the sum of the infinite geometric series was well known in India in the 9th century AD:

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}, \text{ when } r < 1.$$

KINDS OF SERIES

It thus appears that the Hindus studied the arithmetic and geometric series at a very early date. Āryabhaṭa I (499), Brahmagupta (628) and other posterior writers considered also the cases of the sums of the sums, the squares and the cubes of the natural numbers. Mahāvīra (850) gave a rule for the summation of an interesting arithmetico-geometric series, viz.

$$\sum_{1}^{n} t_m$$

where $t_1 = a$ and $t_m = rt_{m-1} \pm b,\ m \geqslant 2$;
and Nārāyaṇa (1356) considered the summation of the figurate numbers of higher orders.

TECHNICAL TERMS

The Sanskrit term for a series is *śreḍhī*, meaning literally "progression", "any set or succession of distinct things", or *śreṇī* (or *śreṇi*), literally "line", "row", "series", "succession"; hence in relation to mathematics it implies "a series or progression of numbers". Thus, it is clear that the modern terms progression and series are analogous to the Hindu terms and they seem to have been adopted in the West under Hindu influence, in preference to the Greek term **εκθεσις (ekthesis)**which literally means a setting forth. The Sanskrit name for a term of the series is *dhana*[8] (literally, "any valued object"). The first term is called *ādi-dhana* ("first term") and any other term *iṣṭa-dhana* ("desired term"). When the series is finite, its last term is called *antya-dhana* ("last term"), and the middle term *madhya-dhana* ("middle term"). Often, for the sake of abridgement, the second words of these compound names are deleted, so that we have the terms *ādi*, *iṣṭa*, *madhya* and *antya* in their places. The first term is also called *prabhava* ("initial term"), *mukha* ("face") or its synonyms. The technical names for the common difference in an arithmetic series are *caya* or *pracaya* (from the root *cay* "to go", hence meaning "that by which the terms goes", that is, "increment"), *uttara* ("difference", "excess"), *vṛḍhi* ("increment"), etc. The common ratio in a geometric series is technically called *guṇa* or *guṇaka* ("multiplier") and so this series is distinguished from the arithmetic series by the specific name *guṇa-śreḍhī*. The number of terms in a series is known as *pada* ("step", meaning "the mumber of steps in the sequence") or *gaccha* ("period"). The sum is called *sarva-dhana* ("total of all terms"), *śreḍhī-phala* ("result of the progression"), *śreḍhī-gaṇita* (or simply *gaṇita*, because the sum of of the series is obtained by computation), and *śreḍhīsaṃkalita* (or in short *saṃkalita*, "sum of the series")

The above-mentioned technical terms occur commonly in almost all the known Hindu treatises on arithmetics from the so-called Bakhshali treatise (*c*. 200) onwards. But in the latter, the series has been designated by *varga*

meaning "group". Occasionally, we meet with the terms *pankti*[9] and *dhārā*[10], which signify "continuous line or series". Nārāyaṇa (1356) has used also a special term, *āya* (literally, "income") for the sum of natural numbers.

SUM OF AN A.P.

Problems on the summation of arithmetic series are met with in the earliest available Hindu work on mathematics, the *Bakhshali Manuscript*. The statement of the formula for the sum begins with the word *rūponā*, so that summation is indicated by the terms *rūponā karanena* ("by the operation *rūponā*, etc.") throughout the work. In the statement of the solution of problems, the first term, the common difference and the number of terms, are written together and the resulting sum after these, as follows:

" *ā* 1	*u* 1	*pa* 19	*rūpoṇā karaṇena phalam*	190 "
1	1	1		1

In the above, *ā* stands for *ādi* ("first term"), *u* for *uttara* ("common difference"), and *pa* for *pada* ("number of terms"). The above quotation may be translated thus: "the first term is $\frac{1}{1}$, the common difference is $\frac{1}{1}$, and the number of terms is $\frac{19}{1}$; therefore, performing *rūponā*, etc. the sum is $\frac{190}{1}$"[11].

Āryabhaṭa I (499) states the formulae for finding the arithmetic mean and the partial sum of a series in A.P. as follows:

"Diminish the given number of terms by one, then divide by 2, then increase by the number of the preceding terms (if any), then multiply by the common difference, and then increase by the first term of the (whole) series: the result is the arithmetic mean (of the given number of terms). This multiplied by the given number of terms is the sum of the given terms. Alternatively, multiply the sum of the first and last terms (of the series or partial series to be summed up) by half the number of terms"[12].

Let the series be

$$a + (a + d) + (a + 2d) + \dots$$

Then the rule says that:

(1) the arithmetic mean of the n terms

$$(a + pd) + (a + \overline{p+1}\,d) + \dots + [a + (p+n-1)\,d]$$
$$= a + (\frac{n-1}{2} + p)\,d;$$

(2) the sum of the n terms

$$(a + pd) + (a + \overline{p+1}\, d) + \ldots + [a + (p+n-1)\, d] = n\,[a + (\frac{n-1}{2} + p)\, d].$$

In particular (when p = 0)

(3) the arithmetic mean of the series

$$a + (a + d) + (a + 2d) + \ldots + [a + (n-1)\, d] = a + \frac{n-1}{2}\, d;$$

(4) the sum of the series

$$a + (a + d) + (a + 2d) + \ldots + [a + (n-1)\, d] = n\,[a + \frac{n-1}{2}\, d].$$

Alternatively, the sum of n terms of an arithmetic series with A as the first term and L as the last term

$$= \frac{n}{2}\,(A + L),$$

where $\frac{1}{2}(A + L)$ is the arithmetic mean of the terms.

Brahmagupta says:

"The last term is equal to the number of terms minus one, multiplied by the common difference, (and then) added to the first term. The arithmetic mean (of the terms) is half the sum of the first and the last terms. This (arithmetic mean) multiplied by the number of terms is the sum"[13]

Similar statements occur in the works of śrīdhara[14], Āryabhaṭa II[15], Bhāskara II[16] and others. Mahāvīra points out that the common difference may be a positive or negative quantity[17].

The particular case

$$\sum_{1}^{n} r = \frac{n\,(n + 1)}{2}$$

is mentioned in all the Hindu works[18]

Ordinary Problems on A.P.

The problems of finding out (1) the first term or (2) the common difference or (3) the number of terms, are common to all Hindu works. They occur first in the *Bakhshali Manuscript*[19]. The problem of finding the number of terms requires the solution of a quadratic equation[20]. Some indeterminate problems in which more than one of the above quantities are unknown also occur in the *Bakhshali Manuscript*, the *Gaṇita-sāra-saṅgraha* of Mahāvīra and the *Gaṇita-kaumudī* of Nārāyaṇa. A typical example of such problems is the finding out of an arithmetic series that will have a given sum and a given number of terms.

As illustrations of some other types of Hindu problems arithmetical progression may be mentioned the following:

(1) There were a number of utpala flowers representable as the sum of a series in arithmetical progression, whereof 2 is the first term and 3 the common difference. A number of women divided those flowers equally among them. Each woman had 8 for her share. How many were the women and how many the flowers?[21]

(2) A person travels with velocities beginning with 4 and increasing successively by the common difference 8. Again, a second person travels with velocities beginning with 10 and increasing successively by the common difference of 2. What is the time of their meeting?[22]

(3) The continued product of the first term, the number of terms and the common difference is 12. If the sum of the series is 10, find it[23].

(4) A man starts with a certain velocity and a certain acceleration per day. After 8 days, another man follows him with a different velocity and an acceleration of 2 per day. They meet twice on the way. After how many days do these meetings occur?[24]

Geometric Series

Mahāvīra gives the formula:

$$S = \frac{a\,(r^n - 1)}{r - 1}$$

for the sum of a geometric series whose first term is a and common ratio r. He says:

"The first term when multiplied by the continued product of the common ratio, taken as many times as the number of terms, gives rise to the *guṇadhana*. And it has to be understood that this *guṇadhana*, when diminished by the first term and (then) divided by the common ratio lessened by 1, becomes the sum of the series in geometrical progression"[25].

The same result is stated by him in the following alternative form:

"In the process of successive halving of the number of terms, put zero or 1 according as the result is even or odd. (Whenever the result is odd subtract 1). Multiply by the common ratio when unity is subtracted and multiply so as to obtain square (when otherwise, i.e., when the half is even). When the result of this (operation) is diminished by 1 and is then multiplied by the first term and (is then) divided by the common ratio lessened by 1, it becomes the sum of the series"[26].

If n be the number of terms and r the common ratio, the first half of the above rules gives r^n. This process of finding the nth power of a number was known to Piṅgala (*c.* 200 BC), and has been used by him to find 2^n. The second half of the rule then gives

$$S = \frac{a\,(r^n - 1)}{r - 1}$$

The above formula for the sum is stated by Pṛthūdakasvāmi[27], Āryabhaṭa II[28] and Bhāskara II[29] in the second form which appears to be the traditional method of stating the result.

Mahāvīra has given rules for finding the first term, common ratio or number of terms, one of these being unknown and the others as well as the sum being given[30].

As illustrations of problems on geometric series may be mentioned the following:

(1) Having first obtained 2 golden coins in a certain city, a man goes on from city to city, earning everywhere three times of what he earned immediately before. Say how much he will make on the eighth day?[31]

(2) When the first term is 3, the number of terms 6 and the sum 4095, what is the value of the common ratio?[32]

(3) The common ratio is 6, the number of terms is 5, and the sum is 3110. What is the first term here?[33]

(4) How many terms are there in a geometric series whose first term is 3, the second ratio is 5 and the sum is 228881835937?[34]

SERIES OF SQUARES

The series whose terms are the squares of natural numbers seems to have attracted attention at a fairly early date in India. The formula

$$\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

occurs in the *Āryabhaṭīya*[35] where it is stated in the following form:

"The sixth part of the product of the three quantities consisting of the number of terms, the number of terms plus 1, and twice the number of terms plus 1, is the sum of the squares."

The formula occurs in all the known Hindu works[36]

Mahāvīra (*GSS*, vi. 298, 299) gives the sum of a series whose terms are the squares of the terms of a given arithmetic series.

Let

$$a + (a + d) + \ldots + (a + \overline{r-1}\, d) + \ldots + (a + \overline{n-1}\, d)$$

be an arithmetic series. Then, according to him,

$$a^2 + (a + d)^2 + \ldots + (a + \overline{r-1}\, d)^2 + \ldots + (a + \overline{n-1}\, d)^2$$

$$= n\left[\left(\frac{2n-1}{6} d^2 + ad\right)(n-1) + a^2\right]$$

$$\text{or } n\left[\frac{(2n-1)(n-1)d^2}{6} + a^2 + (n-1)ad\right].$$

Śridhara[37] and Nārāyana[38] give the above result in the following form:

$$\sum_{1}^{n} (a + \overline{r-1}\, d)^2 = a \sum_{1}^{n} [a + 2(r-1)d] + d^2 \sum_{1}^{n-1} r^2.$$

SERIES OF CUBES

Āryābhaṭa I states the formula giving the sum of the series formed by the cubes of natural numbers as follows:

"The square of the sum of the original series (of natural numbers) is the sum of the cubes"[39].

Thus, according to him,

$$\sum_{1}^{n} r^3 = \left(\sum_{1}^{n} r\right)^2 = \left[\frac{n(n+1)}{2}\right]^2$$

The above formula occurs in all the Hindu works. The general case in which the terms of the series are cubes of the terms of a given arithmetic series, has been treated by Mahāvīra[40].

Let

$$S = \sum_{1}^{n} \alpha_r$$

be an arithmetic series whose first term is a, and common difference d. Then, according to Mahāvīra,

$$\sum_{1}^{n} \alpha_r^3 = d.S^2 \pm Sa\,(a \sim d),$$

according as $a >$ or $< d$.

Śrīdhara[41] and Nārāyaṇa[42] have also given the above result in the same form as Mahāvīra.

SERIES OF SUMS

Let

$$N_n = 1 + 2 + 3 + \ldots + n.$$

Then the series

$$\sum_{1}^{n} N_r$$

formed by taking successively the sums up to 1, 2, 3, ... terms of the series of natural numbers, is given in all the Hindu works[43], beginning with that of Āryabhaṭa I, who says:

"In the case of an *upaciti* which has 1 for the first term and 1 for the common difference between the terms, the product of three terms having the number of terms (n) for the first term and 1 for the common difference, divided by six is the *citighana*. Or, the cube of the number of terms plus 1, minus the cube root of the cube[44], divided by 6[45]."

The above rule states that

$$\sum_{1}^{n} N_r = \frac{n(n+1)(n+2)}{6}$$

$$\text{or} \quad = \frac{(n+1)^3 - (n+1)}{6}.$$

The sum of the series $\sum_{1}^{n} N_r$ has been called by Āryabhaṭa I *citighana* which means "the solid content of a pile in the shape of pyramid on a triangular base". The pyramid is constructed as follows:

Form a triangle with $\sum_{1}^{n} m$ things arranged as below:

```
        0  1
      0  0  2
    0  0  0  3
   ......................
  ............................
 0  0  0  .......  0  0  0  (n-1)
0  0  0  ........  0  0  0  0  n
```

Total $\dfrac{n(n+1)}{2}$

Form a similar triangle with $\sum_{1}^{n-1} m$ things and place it on top of the first, then form another such triangle with $\sum^{n-2} m$ things and place it on top of the

first two. Proceed as above till there is one thing at the top. The figure obtained in this manner will be a pyramid formed of n layers, such that the base layer consists of $\sum_{1}^{n} r$ things, the next higher layer consists of $\sum_{1}^{n-1} r$ things, and so on. The number of things in the solid pyramid *citighana* $= \sum_{1}^{n} N_r$,

where $N_r = \sum_{m=1}^{m=r} m$.

The base of the pyramid is called *upaciti*, so that

$$upaciti = \sum_{m=1}^{m=n} m.$$

The above *citighana* is the series of figurate numbers. The Hindus are known to have obtained the formula for the sum of the series of natural numbers as early as the fifth century BC. It cannot be said with certainty whether the Hindus in those times used the representation of the sum by triangles or not. The subject of piles of shots and other things has been given great importance in the Hindu works, all of which contain a section dealing with *citi* ("piles"). It will not be a matter of surprise if the geometrical representation of figurate numbers is traced to Hindu sources.

MAHĀVĪRA'S SERIES

Mahāvīra (850) has generalized the series of sums in the following manner:

Let $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n$

be a series in arithmetical progression, the first term being α_1, and the common difference β, so that

$$\alpha_r = \alpha_1 + (r - 1)\beta.$$

Mahāvīra considers the following series

$$\sum_{r=1}^{r=n} \left(\sum_{m=1}^{m=\alpha_r} m \right)$$

and gives its sum as

$$\frac{n}{2}\left[\left(\frac{(2n-1)\beta^2}{6} + \frac{\beta}{2} + \alpha_1\beta\right)(n-1) + \alpha_1(\alpha_1+1)\right].$$ [46]

Nārāyaṇa[47] gives the above result in another form, According to him

$$\sum_{r=1}^{r=n}\left(\sum_{m=1}^{m=\alpha_r} m\right) = \left(\sum_{1}^{\alpha_1+\beta} m - \sum_{1}^{\alpha_1} m\right)\sum_{1}^{n-1} m + n\sum_{1}^{\alpha_1} m + \beta^2 \sum_{1}^{n-2}\left(\sum_{1}^{r} m\right).$$

Denoting by N_r the sum of r terms of the series of natural numbers, Nārāyaṇa's result may be written in the form

$$\sum_{r=1}^{r=n} N_{\alpha r} = (N_{\alpha_1+\beta} - N_{\alpha_1})\, N_{n-1} + n\, N_{\alpha_1} + \beta^2 \sum_{1}^{n-2} Nr$$

$$= \left[\frac{(\alpha_1+\beta)(\alpha_1+\beta+1)}{2} - \frac{\alpha_1(\alpha_1+1)}{2}\right]\frac{n(n-1)}{2} + \frac{n_1\,\alpha_1(\alpha_1+1)}{2}$$

$$+ \beta^2\,\frac{(n-2)(n-1)\,n}{6}$$

which can be reduced to Mahāvīra's form.

Śrīdhara[48] puts the result in the form

$$\sum_{r=1}^{r=n}\left(\sum_{m=1}^{m=\alpha_r} m\right) = \frac{1}{2}\left[\sum_{r=1}^{r=n}\alpha_r^2 + \sum_{r=1}^{r=n}\alpha_r\right]$$

NĀRĀYAṆA'S SERIES

Nārāyaṇa has given formulae for the sums of series whose terms are formed successively by taking the partial sums of other series in the following manner:

Let the symbol $^nV_1$ denote the arithmetic series of natural numbers up to n terms; i.e., let

$$'V_1 = 1 + 2 + 3 + \ldots + n.$$

Let $^nV_2$ denote the series formed by taking the partial sums of the series $^nV_1$. Then

$$^{n}V_2 = \sum_{r=1}^{r=n} {}^{r}V_1$$

Similarly, let

$$^{n}V_3 = \sum_{r=1}^{r=n} {}^{r}V_2$$

$$^{n}V_4 = \sum_{r=1}^{r=n} {}^{r}V_3$$

....

$$^{n}V_m = \sum_{r=1}^{r=n} {}^{r}V_{m-1}$$

The series $^{n}V_m$ has been called by Nārāyaṇa as *m-vāra-saṅkalita* ("m-order-series") meaning thereby that the operation of forming a new series by taking the partial sums of a previous series has been repeated m times. The number m may be called the order (*vāra*) of the series.

Nārāyaṇa states the sum $^{n}V_m$ as follows:

"The terms of the sequence beginning with the *pada* (number of terms, i.e., n) and increasing by 1 taken up to the order (*vāra*) plus 1 times are successively the numerators and the terms of the sequence beginning with unity and increasing by 1 are respectively the denominators. The continued product of these (fractions) gives the *vāra-saṅkalita* ("sum of the iterated series of a given order")."

Thus, according to the above, n being the number of terms of the iterated and m the order, we get the following sequence of numbers:

$$\frac{n}{1}, \frac{n+1}{2}, \frac{n+2}{3}, \ldots, \frac{n+m}{m+1}$$

The sum of the series is the continued product of the above sequence, i.e.,

$$^{n}V_m = \frac{n \,.\, (n+1) \,.\, (n+2) \ldots (n+m)}{1 \,.\, 2 \,.\, 3 \ldots (m+1)}$$

Putting m = 1, 2, 3, ..., we get

$$^{n}V_1 = \sum_{r=1}^{n} r = \frac{n\,(n+1)}{1\,.\,2},$$

$$^{n}V_2 = \sum_{r=1}^{n} rv_1 = \frac{n\,(n+1)\,(n+2)}{1\,.\,2\,.\,3},$$

$$^{n}V_3 = \sum_{r=1}^{n} rv_2 = \frac{n\,(n+1)\,(n+2)\,(n+3)}{1\,.\,2\,.\,3\,.\,4}$$

and so on.

Nārāyaṇa (1356) has made use of the numbers of the *vāra-saṅkalita* in the theory of combinations, in chapter xiii of his *Gaṇita-kaumudī*. The series discussed above are now known as the series of figurate numbers. They seem to have been first studied in the west by Pascal (1665).

GENERALISATION

Nārāyaṇa has considered the more general series obtained in the same way as above from a given arithmetical progression.

Let

$$^{n}S_1 = \sum_{1}^{n} \alpha_r = \alpha_1 + \alpha_2 + \ldots + \alpha_n,$$

where $\sum_{1}^{n} \alpha_r$ is an arithmetic series whose first term is α_1 and common difference β. As above, let us define the iterated series $^{n}S_2$, $^{n}S_3$, ... $^{n}S_k$ as follows:

$$^{n}S_2 = \sum_{r=1}^{r=n} {}^{r}S_1$$

$$^{n}S_3 = \sum_{r=1}^{r=n} {}^{r}S_2$$

....

$$^{n}S_k = \sum_{r=1}^{r=n} {}^{r}S_{k-1}$$

Nārāyaṇa states the formula for the sum of the series ${}^{n}S_{r}$ thus:

"The sum of the iterated series of the given order derived from the natural numbers equal to the given number minus 1 is put down at two places. These become the multipliers. The order as increased by unity being divided by the given number of terms as diminished by unity is a multiplier of the first (of these multipliers). The first term and the common difference multiplied respectively by the two quantities and (the results) added together gives the required sum of the iterated series."

Suppose it be required to find ${}^{n}S_{m}$, where n is the *pada* ("number of terms") and m the *vāra* ("order") of the iterated series. Let, as before, ${}^{n}V_{r}$ denote the iterated series of the rth order derived from the series of n natural numbers. Then, taking ${}^{n-1}V_{m}$ as two places, and multiplying the first of these by

$\frac{m+1}{n-1}$ as directed, we get

$$\frac{m+1}{n-1}\,{}^{n-1}V_{m} \text{ and } {}^{n-1}V_{m}$$

Multiplying the first term (α_1) and the common difference (β) by these two respectively and adding we get the required sum

$${}^{n}S_{m} = \alpha_1 \frac{m+1}{n-1}.\,{}^{n-1}V_{m} + \beta\,{}^{n-1}V_{m}.$$

Rationale

The above formula has been evidently obtained by Nārāyaṇa as follows:

$${}^{n}S_{1} = \sum_{1}^{n} \alpha_r = \alpha_1 + (\alpha_1 + \beta) + \dots + (\alpha_1 + \overline{n-1}\,\beta)$$

$$= n\,[\alpha_1 + \frac{n-1}{2}\beta]$$

$${}^{n}S_{2} = \sum_{1}^{n} {}^{r}S_{1} = \alpha_1 \sum_{1}^{n} r + \beta \sum_{1}^{n} \frac{r\,(r-1)}{2}$$

$$= \alpha_1\ {}^{n}V_{1} + \beta\ .\ {}^{n-1}V_{2}$$

$$^{n}S_3 = \sum_{1}^{n} {}^{n}S_2 = \alpha_1 \sum_{1}^{n} {}^{r}V_1 + \beta \sum_{1}^{n} {}^{r-1}V_2$$

$$= \alpha_1 \,.\, {}^{n}V_2 + \beta \quad {}^{n-1}V_3$$

....

$$^{n}S_m = \alpha_1 \,.\, {}^{n}V_{m-1} + \beta \,.\, {}^{n-1}V_m$$

But $^{n}V_{m-1} = \dfrac{m+1}{n-1} \, {}^{n-1}V_m$

$$\therefore \; {}^{n}S_m = \alpha_1 \frac{m+1}{n-1} \, {}^{n-1}V_m + \beta \,.\, {}^{n-1}V_m.$$

NĀRĀYAṆA'S PROBLEM

The above series have been investigated by Nārāyaṇa in order to solve the following type of problems:

"A cow gives birth to one calf every year. The calves become young and themselves begin giving birth to calves when they are three years old. O learned man, tell me the number of progeny produced during twenty years by one cow."

Solution

(i) The number of calves produced during 20 years by the cow is 20.

(ii) The first calf becomes a cow in 3 years and begins giving birth to calves every year, so that the number of its progeny during the period under consideration is $(20-3) = 17$. Similarly, the second calf becoming a cow produces, during the period under consideration $(19-3) = 16$ calves, and so on. The total number of calves of the second generation $= \sum_{1}^{17} r = {}^{17}V_1$.

(iii) The first calf of the eldest cow (of the group of 17) produces during the period under consideration $(17-3) = 14$ calves; the second calf of the same group produces 13 calves; and so on. The total progeny (of the second generation) of the group of 17 in (ii) is

$$14 + 13 + 12 + \ldots + 1 = {}^{14}V_1.$$

Similarly, the total progeny of 16 in (ii) is $^{13}V_1$, of the group of 15 in (ii) is $^{12}V_1$, and so on. Thus, the total progeny of the third generation is

$$\sum_{1}^{14} {}^{r}V_1 = {}^{14}V_2,$$

Similarly, the total progeny of the fourth generation is

$$\sum_{1}^{(14-3)} {}^{r}V_2 = {}^{11}V_3,$$

and so on.

The total number of cows and calves at the end of 20 years is

$$1 + 20 + {}^{17}V_1 + {}^{14}V_2 + {}^{11}V_3 + {}^{8}V_4 + {}^{5}V_5 + {}^{2}V_6$$

$$= 1 + 20 + \frac{17 \cdot 18}{1 \cdot 2} + \frac{14 \cdot 15 \cdot 16}{1 \cdot 2 \cdot 3} + \frac{11 \cdot 12 \cdot 13 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$+ \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$$

$$= 1 + 20 + 153 + 560 + 1001 + 792 + 210 + 8$$

$$= 2745.$$

After giving the solution of the problem Nárāyaṇa remarks:

"An alternative method of solution is by means of the *Meru* used in the theory of combination in connection with (the calculations regarding) metre. This I have given later on."

Miscellaneous Results

The following results have been given by Śrīdhara, Mahāvīra and Nārāyaṇa:

49: $n^2 = 1 + 3 + 5 + \ldots$ to n terms

50: $$n^3 = \sum_{1}^{n} [3r(r-1) + 1]$$

$$= 3 \sum_{1}^{n} r(r - 1) + n$$

51: $n^3 = n + 3n + 5n + \ldots$ to n terms

52: $n^3 = n^2 . (n-1) + \sum_{r=1}^{n} (2r-1)$

53: $[(n+3) \frac{n}{4} + 1] (n^2+n) = \sum_{1}^{n} r + \sum_{1}^{n} r^2 + \sum_{1}^{n} r^3 + \sum_{1}^{n} (\sum_{1}^{r} m)$

$= \sum_{1}^{n} r (1 + r + r^2 + \frac{r+1}{2})$

54: $\sum_{1}^{n} r + n^2 = 3 \sum_{1}^{n} r - n.$

$\sum_{1}^{n} r + n^3 = \frac{(6n+1) (\sum_{1}^{n} r + n^2) + 4n}{9}$

55: $\sum_{1}^{n} r + n^2 + n^3 = \frac{n (n+1) (2n+1)}{2}$

56: $\sum_{m=1}^{m=n} \sum_{r=1}^{r=m} r + \sum_{r=1}^{r=n} r^2 + \sum_{r=1}^{r=n} r^3 = \frac{n (n+1)^2 (n+2)}{4}$

57: $\sum_{r=1}^{a} r + \sum_{r=1}^{a+d} r + \sum_{r=1}^{a+2d} r + \ldots$ to n terms

$= \frac{1}{2} [\sum_{(r=1)}^{r=n} (a + (r-1) d)^2 + \sum_{r=1}^{r=n} (a + (r-1) d)].$

58: $S \pm (\frac{S}{a} - n) \frac{m}{r-1} = a + (ar \pm m) + [(ar \pm m) + m]$

$\pm [(ar \pm m) r \pm m] r \pm m + \ldots$ to n terms.

where $S = a + ar + ar^2 + \ldots$ to n terms.

BINOMIAL SERIES

The development of $(a+b)^n$ for integral values of n has been known in India from very early times. The case $n=2$ was known to the authors of the *Śulba Sūtras* (1500-1000 BC). The series formed by the binomial coefficients

$$^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$$

seems to have been studied at a very early date. Piṅgala (*c.* 200 BC), a writer on metrics, knew the sum of the above series[59] to be 2^n. This result is found also in the works of Mahāvīra[60] (850), Pṛthūdakasvāmī[61] (860) and all later writers.

PASCAL TRIANGLE

The so-called Pascal triangle was known to Piṅgala, who explained the method of formation of the triangle in short aphorisms (*sūtra*). These aphorisms have been explained by the commentator Halāyudha thus:

"Draw one square at the top; below it draw two squares, so that half of each of them lies beyond the former on either side of it. Below them, in the same way, draw three squares; then below them four; and so on up to as many rows as are desired: this is the preliminary representation of the *Meru*. Then putting down 1 in the first square, the figuring should be started. In the next two squares put 1 in each. In the third row put 1 each of the extreme squares, and in the middle square put the sum of the two numbers in the two squares of the second row. In the fourth row put 1 in each of the two extrême squares: in an intermediate square put the sum of the numbers in the two squares of the previous row which lie just above it. Putting down of numbers in the other rows should be carried on in the same way. Now the numbers in the second row of squares show the monosyllabic forms: there are two forms, one consisting of one long and the other one short syllable. The numbers in the third row give the disyllabic forms: in one form all syllables are long, in two forms one syllable is short (and the other long), and in one all syllables are short. In this row of the squares we get the number of variations of the even verse. The numbers in the fourth row of squares represent trisyllabic forms. There one form has all syllables long, three have one syllable short, three have two short syllables, and one has all syllables short. And so on in the fifth and succeeding rows; the figure in the first square gives the number of forms with all syllables long, that in the last all syllables short, and the figures in the successive intermediate squares represent the number of forms with one, two, etc. short syllables".

Thus, according to the above, the number of variations of a metre containing n syllables will be obtained from the representation of the *Meru* as follows:

Number of syllables								Total number or variations
	1							
1...	1	1						... 2 = 2^1
2...	1	2	1					... 4 = 2^2
3...	1	3	3	1				... 8 = 2^3
4...	1	4	6	4	1			... 16 = 2^4
5...	1	5	10	10	5	1		... 32 = 2^5
6...	1	6	15	20	15	6	1	... 64 = 2^6

Meru Prastāra

From the above it is clear that Piṅgala knew the result

$$^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_{n-1} + {}^nC_n = 2^n.$$

INFINITE SERIES

Early History

As already remarked, the formula for the sum of an infinite geometric series, with common ratio less than unity, was known to Jain mathematicians of the ninth century. Application of this formula was made to find the volume of the frustum of a cone in Vīrasena's commentary on the ***Ṣṭkhaṇḍāgama,*** which was completed about 816 AD. The mathematicians of South India, especially those of Kerala, seem to have made notable contribution to the theory of infinite series. We find that in the first half of the fifteenth century they discovered what is now known as Gregory's series. Use of this series seems to have been made for the calculation of π, and in astronomy. As the works of this period are not available to us, it is not possible to trace the gradual evolution of the infinite series in India. Some of these series that are found to occur in the works of the Kerala mathematicians of the 16th, 17th and 18th centuries are given below.

Series for the Arc of a Circle

Śaṅkara Vāriyar (1500-1560), the commentator of Nīlakaṇṭa Somayāji's *Tantra-saṅgraha*, gives an infinite series for the arc of a circle in terms of its sine and cosine and the radius of the circle. He says:

"By the method stated before for the calculation of the circle, the arc corresponding to a given value of the sine can be found. Multiply the given

value (*iṣṭa*) of the sine (*jyā*) by the radius and divide by the cosine (*koṭijyā*). The result thus obtained is the first quotient. Then operating again and again with the square of the (given) sine as the multiplier and the square of the cosine as the divisor, obtain from the first quotient, other quotients. Divide the successive quotients by the odd numbers 1, 3, etc., respectively. Now subtract the even order of quotients from the odd ones. The remainder is the arc (required)"[62].

That is to say, if R denotes the radius of a circle, α an arc of it and θ the angle subtended at the centre by that arc, then

$$R\theta = \alpha = \frac{R \sin \theta}{1 \quad \cos \theta} - \frac{R \sin^3 \theta}{3 \,.\, \cos^3 \theta} + \frac{R \sin^5 \theta}{5 \,.\, \cos^5 \theta} - \frac{R \sin^7 \theta}{7 \,.\, \cos^7 \theta} + \dots$$

This series will be convergent if $\sin \theta < \cos \theta$, that is, if $\theta < \pi/4$. But if $\theta > \pi/4$, the series will be divergent and so the rule appears to fail. If in that case, however, we take $\sin (\pi/2 - \theta)$ as given instead of $\sin \theta$, then in accordance with the rule, we shall get the series

$$\frac{R\pi}{2} - \alpha = \frac{R \sin (\pi/2-\theta)}{1 \,.\, \cos (\pi/2-\theta)} - \frac{R \sin^3 (\pi/2-\theta)}{3 \,.\, \cos^3 (\pi/2-\theta)} + \frac{R \sin^5 (\pi/2-\theta)}{5 \,.\, \cos^5 (\pi/2-\theta)} - \dots$$

$$\text{or } \frac{R\pi}{2} - \alpha = \frac{R \cos \theta}{1 \,.\, \sin \theta} - \frac{R \cos^3 \theta}{3 \,.\, \sin^3 \theta} + \frac{R \cos^5 \theta}{5 \,.\, \sin^5 \theta} - \dots$$

which is convergent. Knowing the value of $R\pi/2 - \alpha$, we can easily calculate the value of α. Thus, the rule will give the desired result even in the case $\theta > \pi/4$. Hence, the author remarks:

"Of the arc and its complement, one should take here (the sine of) the smaller as given (*iṣṭa*): this is what has been stated"[63].

The above series is stated also by Puthumana Somayāji (*c*. 1660-1740) and Śaṅkaravarman (1800-38). The former writes:

"Find the first quotient by dividing by the cosine the given sine as multiplied by the radius. Then get the other quotients by multiplying the first and those successively resulting by the square of the sine and dividing them in the same way by the square of the cosine. Now dividing these quotients respectively by 1, 3, 5, etc., subtract the sum of even ones (in the series) from the sum of the odd ones. Thus, the sine will become the arc"[64].

Śaṅkaravarman says:

"Divide the product of the radius and the sine by the cosine. Divide this quotient and others resulting successively from it on repeated multiplication by the square of the sine and division by the square of the cosine by 1, 3, 5, etc., respectively. Then subtract the sum of the even quotients (in the series) from the sum of the odd ones. The remainder is the arc (required)"[65]

Introducing the modern tangent function, the above series can be written as

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \frac{1}{7}\tan^7\theta + \ldots$$

This series was rediscovered by James Gregory in 1671 and then by G.W. Leibnitz in 1673. It is now generally ascribed to the former. But rightly speaking, this series was first discovered in India, probably by the Kerala mathematician Mādhava, who lived about 1340-1425 AD.

For the case $\theta = \pi/4$, Jyeṣṭhadeva (*c.* 1500-*c.* 1610), in his *Yuktibhāṣā*, gives three successively better approximations to $\pi/4$[66]

(1) $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \pm \frac{1}{n} \pm \frac{1}{n+1}$$

(2) $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \pm \frac{1}{n} \pm \frac{{}^1/_2\,(n+1)}{(n+1)^2 + 1}$$

(3) $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \pm \frac{1}{n} \pm \frac{[{}^1/_2\,(n+1)]^2 + 1]}{{}^1/_2\,(n+1)\,[(n+1)^2 + 4 + 1]}$$

Śaṅkara Vāriyar has also stated (2)[67] and (3)[68] and in addition the approximation[69]

$$\frac{1}{2} + \frac{1}{2^2-1} - \frac{1}{4^2-1} + \ldots \pm \frac{1}{n^2-1} \pm \frac{1}{2[(n+1)^2 + 2]}.$$

A number of infinite series expansions for π (circumference/diameter) occur in the works of Śaṅkara Vāriyar, Puthumana Somayaji and Śaṅkaravarman. Some of these are:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$$

70: $$\pi = \sqrt{12}\,[\frac{1}{9(2.1-1)} + \frac{1}{9^2(2.3-1)} + \frac{1}{9^3(2.5-1)} + \ldots]$$

$$-\frac{\sqrt{12}}{3}\left[\frac{1}{9(2.2-1)}+\frac{1}{9^2(2.4-1)}+\frac{1}{9^3(2.6-1)}+\ldots\right].$$

71: $$\pi=\sqrt{12}\left[1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.3^3}+\ldots\right]$$

72: $$\pi=3+4\left[\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\ldots\right]$$

73: $$\pi=16\left[\frac{1}{1^5+4.1}-\frac{1}{3^5+4.3}+\frac{1}{5^5+4.5}-\ldots\right]$$

74: $$\pi=8\left[\frac{1}{2^2-1}+\frac{1}{6^2-1}+\frac{1}{10^2-1}+\ldots\right]$$

75: $$\pi=4-8\left[\frac{1}{4^2-1}+\frac{1}{8^2-1}+\ldots\right]$$

76: $$\pi=3+6\left[\frac{1}{(2.2^2-1)^2-2^2}+\frac{1}{(2.4^2-1)^2-4^2}+\frac{1}{(2.6^2-1)^2-6^2}+\ldots\right]$$

Series for the Sine and Cosine of an Arc

The Hindus discovered series also for the sine and cosine of an angle in powers of its circular measure. Puthumana writes:

"In the series of quotients obtained by dividing an arc of a circle severally by 2, 3, etc., times the radius, multiply the arc by the first (term); the resulting product by the second (term); this product again by the third (term); and so on. Put down the even terms of the sequence so obtained after the arc and the odd ones after the radius, and subtract the alternative ones. The remainders will respectively be the *Jyā* and *Kojyā* of that arc"[77].

That is to say,

$$Jy\bar{a}\ \alpha=\alpha-\frac{\alpha^3}{3!R^2}+\frac{\alpha^5}{5!R^4}-\frac{6\alpha^7}{7!R^6}-\ldots..$$

$$Kojy\bar{a}\ \alpha = R - \frac{\alpha^2}{2\,!\,R} + \frac{\alpha^4}{4\,!\,R^3} - \frac{\alpha^6}{6\,!\,R^5} + \ldots$$

corresponding to our modern series

$$\sin\theta = \theta - \frac{\theta^3}{3\,!} + \frac{\theta^5}{5\,!} - \frac{\theta^7}{7\,!} + \ldots$$

$$\cos\theta = 1 - \frac{\theta^2}{2\,!} + \frac{\theta^4}{4\,!} - \frac{\theta^6}{6\,!} + \ldots$$

These series reappear in the works of Śaṅkaravarman[78]. When θ is small, we have the approximation

$$\sin\theta = \theta - \frac{1}{6}\theta^3$$

Similarly

$$\theta = \sin\theta + \frac{1}{6}\sin^3\theta.$$

Thus, Puthumana says:

"A small arc being diminished by the sixth part of its cube as divided by the square of the radius becomes the *Jyā*. A small *Jyā* being increased in the same way becomes the arc"[79].

So does Śaṅkaravarman[80].

Śaṅkara Vāryar has also given an infinite series expansion for $\sin^2\theta$. He says:

"(Repeatedly) multiply the square of the arc by the square of the arc and divide successively by the square of the radius as multiplied by the squares of 2, etc. diminished by half of their square roots. Write the square of the arc, and below it the successive results and (then starting from the lowest) subtract the lower from that above it. What is thus obtained is the square of the *Jyā*[81]. That is to say,

$$Jy\bar{a}^2\ \alpha = \alpha^2 - \frac{\alpha^4}{(2^2 - 2/2)\ R^2} + \frac{\alpha^6}{(2^2 - 2/2)\ (3^2 - 3/2)R^4}$$

$$- \frac{\alpha^8}{(2^2 - 2/2)\ (3^2 - 3/2)\ (4^2 - 4/2)\ R^6} + \ldots$$

or, in modern notation,

$$\sin^2 \theta = \theta^2 - \frac{\theta^4}{(2^2 - 2/2)} + \frac{\theta^6}{(2^2 - 2/2)\ (3^2 - 3/2)}$$

$$- \frac{\theta^8}{(2^2 - 2/2)\ (3^2 - 3/2)\ (4^2 - 4/2)} + \ldots$$

Notes & References

1. In the *Akmes Papyrus*. Cf. Peet, *Rhind Papyrus*, p.78; Smith, *History*, II, p.498.
2. *TS*, vii. 2.12-17; iv. 3.10.
3. *VS*, xvii. 24.25.
4. xviii. 3. Compare also *Lāṭyāyana Śrauta-sūtra*, viii. 10.1 et seq.; *Kātyāyana Śrauta-sūtra*, xxii. 9. 1-6.
5. T.W. Rhys Davids, *Dialogues of the Buddha*, III, 1921. pp. 70-72.
6. *Bṛhaddevatā* edited in original Sanskrit with English translation by A. Macdonell, Harvard, 1904.
7. 1.3.2. Also see A.N. Singh, *History of India from Jaina Sources*, JA, Vol. xvi, Dec. 1950, No. 2 pp. 54-69.
8. In mathematics *dhana* means an affirmative quantity or plus. This probably explains the use of this term to denote the elements of a series which have to be summed up.
9. See Ch. xiii of the *Gaṇita-kaumudī* of Nārāyaṇa.
10. For instance, see the *Triloka-sāra* of Nemicandra (*c*. 975).
11. The denominator 1 is written in the case of all the integral quantities. This is to show that the quantities involved may have non-integral values also.
12. *Ā*, ii. 19. The commentator Bhāskara I says that several formulae are set out here. For details see *Āryabhaṭīya*, edited and translated by K.S. Shukla in collaboration with K.V. Sarma, New Delhi (1976), pp. 62-63.
13. *BrSpSi*, xii. 17.
14. *TriŚ*, p. 28.
15. *MSi*, xv. 47.
16. *L*, p. 27.
17. *GSS*, p. 102, (290).
18. It is sometimes mentioned in connection with addition, as in Śrīdhara's *Triśatikā* and Mahāvīra's *Gaṇita-sāra-saṅgraha*.
19. See p. 25; p. 35 problem 9; and p. 36 problem 10. The solution of this problem is incorrectly printed.

20. For the equation and its solution see the section on quadratic equation in the chapter on Algebra in Part II.
21. *GSS*, vi. 295.
22. *GSS*, vi. $323^1/2$. A problem of the above type in which one of the men travels with a constant velocity occurs in the *Bakhshali Manuscript*, p. 37.
23. *GK*, *Śreḍhī-vyavahāra*, Ex. under Rule 6.
24. *Ibid*, under Rule 9.
25. *GSS*, ii. 93.
26. *GSS*, ii. 94; also vi. $311^1/2$, where the rule is applied to the case in which the common ratio is a fraction.
27. BrSpSi, xii. 17, quoted in the commentary.
28. *MSi*, xv. 52-53.
29. *L*, p. 31.
30. *GSS*, ii. 97-103.
31. *GSS*, ii. 96.
32. *GSS*, ii. 102 (first half).
33. *GSS*, ii. 102 (second half).
34. *GSS*, ii. 105 (last half).
35. *Ā*, ii. 22.
36. Although this rule does not occur in the *Triśatikā*, it occurs in Śrīdhara's bigger work of which the *Triśatikā* is an abridgement. See *PG*, Rule 102.
37. *PG*, Rule 105.
38. *GK*, *Śreḍhi-vyavahāra*, $17^1/2$ and the first half of 18.
39. *Ā*, ii. 22.
40. *GSS*, vi. 303.
41. *PG*, Rule 107.
42. *GK*, *Śreḍhī-vyavahāra*, 18 (c-d) f.
43. This rule does not occur in the *Triśatikā* of Śrīdhara, but it occurs in his *Pāṭīgaṇita*. See *PG*, Rule 103.
44. This means $[(n+1)^3]^{1/3} = (n+1)$. Recourse is taken to this form of expression for the sake of meter.
45. *Ā*, ii. 21.
46. *GSS*, vi. 305-$305^1/2$.
47. *GK*, I, p. 117, lines 11-16.
48. See *PG*, Rule 106.
49. *Triś*, p. 5; *GSS*, ii. 29; *GK*, i. 18.
50. *Triś*, p. 6; *GSS*, ii. 45; *GK*, i. 22.
51. *GSS*, ii. 44; *GK*, *Śreḍhī-vyavahāra*, 10-11.
52. *Ibid*.

53. *GSS*, vii. 309½.

54. (6) and (7) are given by Nārāyaṇa, *GK*, *l.c.*, Rules 11 and 12.

55. *PG*, Rule 102; *GK*, l.c. Rule 13 (a-b).

56. *PG*, Rule 104.

57. *PG*, Rule 106.

58. *GSS*, vi. 314.

59. Piṅgala, *Chandaḥ Sūtra*, viii.23-27.

60. *GSS*, ii. 94.

61. *BrSpSi*, xii. 17 commentary.

62. Verses 206-208 of Śaṅkara Vāriyar's larger commentary on *TS*, (= *Tantra-saṅgraha)* entitled *Yuktidīpikā*, ed. by K.V. Sarma, Hoshiarpur (1977).

63. Verse 209 (a-b) of the commentary *Yuktidīpikā* on *TS*, ii.

64. *Karaṇapaddhati, vi.* 18.

65. *Sadratnamālā*, iv. 11.

66. C.T. Rajgopal and M.S. Rangachari, "On the Untapped Source of Medieval Keralese Mathematics", *Archives for History of Exact Sciences*, Vol. 18 No. 2, 1978.

67. *Tantrasaṅgrahavyākhyā Yuktidīpikā*, vss. 271-274.

68. *Ibid*, vss. 295-296.

69. *Ibid*, vs. 292.

70. *Sadratnamālā*, iv.1.

71. *Ibid*, iv. 2.

72. *Karaṇapaddhati*, vi. 2.

73. *Tantrasaṅgrahavyākhyā Yuktidīpikā*, vss. 287-288.

74. *Ibid*, vss. 293-294.

75. *Ibid*, vss. 293-294.

76. *Karaṇapaddhati*, vi. 4.

77. *Karaṇapaddhati*, vi. 12f.

78. *Sadratnamālā*, iv. 5.

79. *Karaṇapaddhati*, vi. 19.

80. *Sadratnamālā*, iv. 12.

81. *Tantrasaṅgrahavyākhyā Yuktidīpikā*, vss. 455-456.

Indian Journal of History of Science, **28**(3), 1993

SURDS IN HINDU MATHEMATICS

BIBHUTIBHUSAN DATTA AND AWADHESH NARAYAN SINGH
(Revised by Kripa Shankar Shukla)
"Argara", Hussainganj Crossing, Behind Bata Shoe Co.,
Lucknow-226001

(Received 24 December 1991)

Elementary treatment of surds, particularly their addition, multiplication and rationalization, is found in the *Śulbasūtras*. Fuller treatment of this subject occurs in the works on Hindu algebra where rules for addition and subtraction multiplication and involution, separation and extraction of square-root of surds and compound surds are given. The present article gives an account of the treatment of surds in Hindu mathematics.

The Sanskrit term for the surd is *karaṇī*. Śrīpatī (1039) defines it as follows: "The number whose square-root cannot be obtained (exactly) is said to form an irrational quantity (*karaṇī*)"[1]. Similar definitions are given by Narayaṇa (1356) and others[2]. Of course, the number is to be considered a surd when the business is with its square root. A surd number is indicated by putting down the tachygraphic abbreviation *ka* before the number affected. Thus, *ka* 8 means $\sqrt{8}$ and *ka* 450 means $\sqrt{450}$

ORIGIN OF SURDS

The origin of the term *karaṇī* is interesting. Literally it means "making one" or "producing one". It seems to have been originally employed to denote the cord used for measuring (the side of) a square. It then meant the side of any square and was so called because it made a square (*caturaśra-karaṇī*). Hence, it came to denote the square-root of any number. As late as the second century of the Christian era, Umāsvāti (*c.* 150) treated the terms *mūla* ("root") and *karaṇī* as synonymous. In later times, however, the application of the term has been particularly restricted to its present significance as a surd. Nemicandra (*c.* 975)[3] has occasionally used the generic term *mūla* to signify a surd, e.g., *daśa mūla* = $\sqrt{10}$.

ADDITION AND SUBTRACTION

For addition of surds, we have the following ancient rule:

"Reducing them by some (suitable) number, add the square-roots of the quotients; the square of the result multiplied by the reducer should be known as the sum of the surds". We do not know the name of the author of this rule. It is found to have been quoted by Bhāskara I (629) in his commentary on the *Āryabhaṭīya* (ii. 10). A similar rule is given by Brahmagupta (628):

"The surds being divided by a (suitable) optional number, the square of the sum of the square-roots of the quotients should be multiplied by that optional number (in case of addition); and the square of the difference (of the square-roots of the quotients being so treated will give the difference of the surds)"[4].

Mahāvīra says:

"After reducing (the surd quantities) by an optional divisor, the square of the sum or difference of the square-roots of the quotients is multiplied by the optional divisor, the square-root (of the product) is the sum or difference of the square-root quantities. Know this to be the calculation of surds"[5].

Śrīpati writes:

"For addition or subtraction, the surds should be multiplied (by an optional number) intelligently (selected), so that they become squares. The square of the sum, or difference of their roots, should then be divided by that optional multiplier. Those surds which do not become squares on multiplication (by an optional number), should be put together (side by side)"[6].

Bhāskara II says:

"Suppose the sum of the two numbers of the surds $(a+b)$ as the *mahatī* ('greater') and twice the square-root of their product $(2\sqrt{ab})$ as the *laghu* ('lesser'). The addition or subtraction of these like integers is so (of the original quantities).

Multiply and divide as if a square number by a square number. In addition and subtraction, the square-root of the quotient of the greater surd number divided by the smaller surd number should be increased or diminished by unity; the result multiplied by itself should be multiplied by the smaller surd number. The product (is the sum or difference of the two surds). If there be no (rational) root, the surds should be stated separately side by side"[7].

Nārāyaṇa says:

"Divide the two surds separately by the smaller or greater among them; add or subtract the square-roots of the quotients; then multiply the square of the result by that divisor. The product is the sum or difference.

"Or multiply the two surds by the smaller or greater one among them; add or subtract the square-roots of the products; then dividing the square of the result by that selected multiplier, the quotient is the sum or difference.

"Or divide the greater surd by the smaller one; add unity to or subtract unity from the square-root of the quotient; then multiply the result by itself and

also by the smaller quantity. The result is the sum or difference (required). Or proceed in the same way with the greater surd.

"Or add twice the square-root of the product of the two surds, supposed as if rational, to or subtract that from their sum. The result is the sum or difference. If there be no rational root of the product, then the two surds should be stated severally.

"To add up several surds, divide them by an optional number and then take the sum of the square-root of the quotients. This sum multiplied by itself and also by that divisor will give the sum of them"[8].

Thus, we have the following methods for addition or subtraction of surds:

(i) $\sqrt{a} \pm \sqrt{b} = \sqrt{b}\,(\sqrt{a/b} \pm 1)^2$

(ii) $\sqrt{a} \pm \sqrt{b} = \sqrt{1/a}\,(a \pm \sqrt{ab})^2$

(iii) $\sqrt{a} \pm \sqrt{b} = \sqrt{c}\,(\sqrt{a/c} \pm \sqrt{b/c})^2$

(iv) $\sqrt{a} \pm \sqrt{b} = \sqrt{1/c}\,(\sqrt{ac} \pm \sqrt{bc})^2$

(v) $\sqrt{a} \pm \sqrt{b} = \sqrt{(a + b)} \pm 2\sqrt{ab}$

The optional number c is so chosen that (ac, bc) or (a/c, b/c) become perfect squares.

Brahmagupta and Mahāvīra teach the method (iii), Śrīpati gives (iv). Bhāskara states (i) and (v). Nārāyaṇa gives (i), (ii), (iii) and (v).

Multiplication and Involution

For the multiplication of surd expressions, the Hindu works give an algebraic method. Thus, Brahmagupta says:

"Put down the multiplicand horizontally below itself as many times as there are terms in the multiplier; then multiplying by the *Khaṇḍaguṇana* method (i.e., by the method of multiplication by component parts), add the (partial) products"[9].

Thus, to multiply $\sqrt{a} + \sqrt{b}$ by $\sqrt{c} + \sqrt{d}$, one should proceed as follows:

$$(\sqrt{a} + \sqrt{b})(\sqrt{c} + \sqrt{d}) = (\sqrt{a} + \sqrt{b}) \times \sqrt{c} + (\sqrt{a} + \sqrt{b}) \times \sqrt{d}$$
$$= \sqrt{ac} + \sqrt{bc} + \sqrt{ad} + \sqrt{bd}$$

"The squaring of a surd is (finding) the product of two equal (surds)"[10].

Sripati writes:

"Putting down the multiplicand and multiplier in the manner of the Kapaṭa-sandhi multiply according to the method taught before. But those surds should be added, as before, in which the product yields a perfect square"[11].

"On multiplying two equal surd quantities, the square of that surd is obtained"[12].

Bhāskara II (1150) observes: "For abridgement, multiplication or division of surd expressions should be proceeded with after addition (or subtraction) of two or more terms of the multiplier and multiplicand or of the divisor and dividend"[13].

A similar remark has been made by Nārāyaṇa[14].

Division

Brahmagupta (628) teaches the following method of division of surds:

"Multiply the dividend and divisor separately by the divisor after making an optional term of it negative; then add up the terms. (Do this repeatedly until the divisor is reduced to a single term. Then divide the (modified) dividend by the divisor reduced to a single term"[15].

Śrīpati (1039) writes:

"Reversing the sign, negative or positive, of one of the surds occurring in the denominator, multiply by it both the numerator and the denominator separately and then add together the terms of the (respective) products. Repeat (the operations) until there is left only a single surd in the denominator. By it divide the dividend above. Such is the method of division of surds"[16].

This rule has been almost reproduced by Bhāskara II[17] (1150) and Nārāyaṇa[18] (1356). The latter delivers also another method similar to the division of one algebraic expression by another. He says:

"Multiply the divisor surd so as to make all or some of its terms square such that the sum of their square-roots will be equal to the rational term (in the dividend). Thus will be determined the multiplier surd. Subtract from the dividend the divisor multiplied by that. If there be left a remainder, the sum of the terms of the divisor multiplied by that multiplier should be subtracted from the terms of the dividend. In case of absence of a rational term (in the dividend), that by which the divisor is multiplied and then subtracted for the dividend so as to leave no remainder, will be the quotient"[19].

Example from Bhaskara II[20]:

Divide $\sqrt{9} + \sqrt{450} + \sqrt{95} + \sqrt{45}$ by $\sqrt{25} + \sqrt{3}$

$$\frac{\sqrt{9} + \sqrt{450} + \sqrt{95} + \sqrt{45}}{\sqrt{25} + \sqrt{3}} = \frac{(\sqrt{9} + \sqrt{450} + \sqrt{95} + \sqrt{45})(\sqrt{25} - \sqrt{3})}{(\sqrt{25} + \sqrt{3})(\sqrt{25} - \sqrt{3})}$$

$$= \frac{\sqrt{8712} + \sqrt{1452}}{\sqrt{484}}$$

$$= \sqrt{18} + \sqrt{3}$$

Example from Narayana[21]:

Divide $5 + \sqrt{90} + \sqrt{180} + \sqrt{648}$ by $\sqrt{5} + \sqrt{36}$.

First method:

$$\begin{array}{l|l|l} \sqrt{5} + \sqrt{36}\ ) & 5 + \sqrt{90} + \sqrt{180} + \sqrt{648} & (\ \sqrt{5} + \sqrt{18} \\ & 5 \qquad\quad + \sqrt{180} & \\ \hline & \quad \sqrt{90} + \sqrt{648} & \\ & \quad \sqrt{90} + \sqrt{648} & \end{array}$$

Second method:

$$\frac{\sqrt{175} + \sqrt{150} + \sqrt{105} + \sqrt{90} + \sqrt{70} + \sqrt{60}}{\sqrt{5} + \sqrt{3} + \sqrt{2}}$$

$$= \frac{(\sqrt{175} + \sqrt{150} + \sqrt{105} + \sqrt{90} + \sqrt{70} + \sqrt{60})(\sqrt{5} + \sqrt{3} - \sqrt{2})}{(\sqrt{5} + \sqrt{3} + \sqrt{2})(\sqrt{5} + \sqrt{3} - \sqrt{2})}$$

$$= \frac{\sqrt{2100} + \sqrt{1800} + \sqrt{1260} + \sqrt{1080}}{\sqrt{60} + \sqrt{36}}$$

$$= \frac{(\sqrt{2100} + \sqrt{1800} + \sqrt{1260} + \sqrt{1080})(\sqrt{60} - \sqrt{36})}{(\sqrt{60} + \sqrt{36})(\sqrt{60} - \sqrt{36})}$$

$$= \frac{\sqrt{20160} + \sqrt{17280}}{\sqrt{576}}$$

$$= \sqrt{35} + \sqrt{30}$$

RULE OF SEPARATION

Bhāskara II gives a rule for an operation converse to that of addition. He says:

"(Find) a square number by which the compound-surd will be exactly divisible. Breaking up the square-root of that (square-number) into parts at pleasure, multiply the square of the parts of the previous quotient. These will be the several component surds[22]. A similar rule is stated by Nārāyaṇa:

"Divide the compound-surd by the square of some number so as to leave no remainder. Parts of it multiplied by themselves and also by the quotient will be the (component) terms of the surd"[23].

That is to say, if $N = m^2k$ and $m = a + b + c + d$, then

$$\sqrt{N} = \sqrt{m^2k} = m\sqrt{k} = (a + b + c + d)\sqrt{k},$$

$$= (\sqrt{a^2k} + \sqrt{b^2k} + \sqrt{c^2k} + \sqrt{d^2k}).$$

EXTRACTION OF SQUARE-ROOT

For the extraction of the square-root of a surd expression, Brahmagupta described the following method:

"The optionally chosen surds being subtracted from the square of the absolute (i.e., rational) term, the square-root of the remainder should be added to and subtracted from the rational term and halved; then the first is considered as a rational term and the second a surd different from the previous. (Such operations should be carried on) repeatedly (if necessary)"[24].

Illustrative example from Pṛthūdakasvāmi (860):

To find the square-root of $16 + \sqrt{120} + \sqrt{72} + \sqrt{60} + \sqrt{48} + \sqrt{40} + \sqrt{24}$. It has been solved substantially as follows:

Subtract the surd numbers 120, 72, 48 from the square of the rational number, viz., 256; the remainder is $(256-120-72-48) = 16$. Its root is 4; $\frac{1}{2}(16 \pm 4) = 10, 6$. Now subtracting the surd numbers 60 and 24 from 10^2, we get 16; its root is 4; $\frac{1}{2}(10 \pm 4) = 7, 3$. Again subtracting the surd number 40 from 7^2, we have 9; its root is 3; and $\frac{1}{2}(7 \pm 3) = 5, 2$. Hence, the required square-root is $\sqrt{6} + \sqrt{5} + \sqrt{3} + \sqrt{2}$.

The same method is taught by Śrīpati (c. 1039)[25] and Bhāskara II (1150). The latter says:

"From the square of the rational number in the (proposed) square-surd, subtract the rational number equivalent to one or more of the surd numbers; the square-root of the remainder should be severally added to and subtracted from the rational number; halves of the results will be the two surds in the square-root. But if there be left any more surd term in the (proposed) square surd, the greater surd number amongst those two should again be regarded as a rational number (and the same operations should be repeated)"[26].

The above exmaple of Pṛthūdakasvāmi is solved by Bhāskara substantially thus:

$$\sqrt{16^2 - (48 + 40 + 24)} = \sqrt{144} = 12; \frac{1}{2}(16 \pm 12) = 14, 2.$$

$$\sqrt{14^2 - (120 + 72)} = 2, \frac{1}{2}(14 \pm 2) = 8, 6.$$

$$\sqrt{8^2 - 60} = 2, \frac{1}{2}(8 \pm 2) = 5, 3$$

Therefore,

$$(16 + \sqrt{120} + \sqrt{72} + \sqrt{60} + \sqrt{48} + \sqrt{40} + \sqrt{24})^{1/2} = \sqrt{6} + \sqrt{5} + \sqrt{3} + \sqrt{2}.$$

For the above rule, all the terms of the surd expression have been contemplated to be positive, as is also clear from the illustrative examples given. For the case in which there is a negative term, Bhāskara II lays down the following procedure:

"If there be a negative surd in the square (expression), the traction of roots should be proceeded with supposing it as if positive; but of the two surds deduced one, chosen at pleasure by the intelligent mathematician, should be taken as negative"[27]

Example from Bhāskara II[28]:

To find the square-root of $10 + \sqrt{24} - \sqrt{40} - \sqrt{60}$.

The solution is given substantially as follows:

$$\sqrt{10^2 - (40 + 60)} = 0, \frac{1}{2}(10 \pm 0) = 5, 5$$

$$\sqrt{5^2 - 24} = 1, \quad \frac{1}{2}(5 \pm 1) = 3, 2$$

Therefore, $(10 + \sqrt{24} - \sqrt{40} - \sqrt{60})^{1/2} = \sqrt{3} + \sqrt{2} - \sqrt{5}$

$$\text{or } \sqrt{10^2 - (24 + 60)} = 4, \quad \frac{1}{2}(10 \pm 4) = 7, 3$$

The greater number, viz., 7 is considered as negative. Then

$$\sqrt{7^2 - 40} = 3, \quad \frac{1}{2}(7 \pm 3) = 5, 2$$

Hence, $(10 + \sqrt{24} - \sqrt{40} - \sqrt{60})^{1/2} = \sqrt{3} + \sqrt{2} - \sqrt{5}$

Also, $(10 + \sqrt{24} - \sqrt{40} - \sqrt{60})^{1/2} = \sqrt{5} - \sqrt{3} - \sqrt{2}$

Limitation of the Method

Bhāskara II indicates how to test whether a given multinomial surd has a square-root at all or not. "This matter has not beeen explained at length", observes he, "by previous writers. I do it for the instruction of the dull"[29]. He then says:

"In a square surd, the number of irrational terms must be equal to a number same as the sum of the (natural) number 1, etc. In a square surd having three irrational terms, the rational number equal to two of the surd numbers; in a square surd having six irrational number terms, the rational equal to three of them in one of ten irrational terms, integers equal to four of them; and in one of fifteen irrational terms, integers equal to five of them; having been subtracted from the square of the rational term the square-root of the remainder should be extracted. If (done) otherwise (in any case), it will not be proper. The numbers to be subtracted from the square of the rational number (in extracting roots of a square-surd) should be exactly divisible by four times the smaller term in the resulting root-surd. The quotients obtained by this exact division will be the surd terms in the root. If they are not obtained by the last rule, then the (resulting) root is wrong[30]"

He has added the following explanatory notes to the above rule: "In the square of an expression containing irrational terms, there must be a rational term. In the square of (an expression consisting of) a single surd, there will be only a ratio term; of two surds, one surd together with a rational term; of three surds, three irrational terms and a rational term; of four surds, six; of five surds, ten; of six surds, fifteen; and so on. Thus, in the square of surd expressions consisting of two or more irrational terms, the number of irrational terms will be

equal to the sum of the natural numbers one, etc. respectively, besides the rational term. So if in an example (proposed), the number (of irrational terms present) be not such; then it must be considered as a compound surd. Break it up (into required number of component surds) and then extract the square-root. This is what has been implied. Thus will be clear the significance of the rule, "In a square surd having three irrational terms, the rational number equal to two of the surd numbers, etc."

Illustrative examples with solution from Bhāskara II[30]:

Example 1: Find the square root of $10 + \sqrt{32} + \sqrt{24} + \sqrt{8}$.

"In this square there being three surd terms, a rational number equivalent to two of the surd numbers is first subtracted from the square of the rational term and the root (of the remainder) extracted. Then proceeding in the same way with (the remaining) one term, no root is found in this case. Hence this (i.e., the proposed expression) does not possess a root expressible in surd terms. If, however, we extract the root by subtracting, contrary to the rule, an integer equivalent to all the surd terms, we get $\sqrt{2} + \sqrt{8}$. But this is wrong as its square is 18.

"Or on adding together the surds $\sqrt{32}$ and $\sqrt{8}$, (the expression becomes) $10 + \sqrt{72} + \sqrt{24}$. Then (by the rule) we obtain $2 + \sqrt{6}$. But that is also erroneous."[31].

Example 2: Find the square-root of $10 + \sqrt{60} + \sqrt{52} + \sqrt{12}$.

"Here in this square, are present three surd terms; so subtracting a rational number equal to two surd numbers, viz. 52 and 12, the two surd terms for the root are obtained as $\sqrt{8}$ and $\sqrt{2}$; of these the smaller one, namely, 2 multiplied by four, that is 8, does not exactly divide 52 and 12. So they should not be subtracted, for it has been stated, 'The numbers to be subtracted from the square of the rational number (in extracting root of a square surd) should be exactly divisible by four times the smaller term in the resulting root-surd'. Let it, however, be supposed that the mention of 'the smaller term' here is metaphorical and may sometimes imply also 'the greater term' and that it should be considered as 'the greater term', if with that root-surd as the rational term other surd terms are deducible. Now on doing so we obtain for the root $\sqrt{2} + \sqrt{3} + \sqrt{5}$. But this is also wrong; for its square is $10 + \sqrt{24} + \sqrt{40} + \sqrt{60}$"[32].

Example 3: Extract the root of $13 + \sqrt{48} + \sqrt{60} + \sqrt{20} + \sqrt{44} + \sqrt{32} + \sqrt{24}$.

"There being six surd terms in this, an integer equal to three of the surd terms should be first subtracted from the square of the rational term and the root (of the remainder) found; next an integer equal to two of the surd terms and then

an integer equal to one surd term (should be subtracted). But on so doing, no root is found in this instance. If we, however, proceed in a different way and subtract from the square of the rational term first an integer equal to the first surd term, then an integer equal to the second and third terms and lastly an integer equal to the remaining surd terms, we get for the root $\sqrt{1} + \sqrt{2} + \sqrt{5} + \sqrt{5}$. But this is incorrect, since its square is $13 + \sqrt{8} + \sqrt{80} + \sqrt{160}$[33]".

Bhāskara then observes in general: "This is certainly a defect of those (ancient writers) who have not defined the limitations of this method ~~of~~ extracting the square root of a surd. In case of such square surds, the roots should be found by taking the roots of the surd terms by the method for finding the approximate values of the roots and then combining them with the rational term[34]"

Further he says: The mention of 'the greater surd' is metaphorical, for sometimes it might imply the less"[34].

Example from Bhāskara II[35] (1150):

To find the root of $17 + \sqrt{40} + \sqrt{80} + \sqrt{200}$

Here

$$\sqrt{17^2 - (80 + 200)} = 3, \quad \frac{1}{2}(17 \pm 3) = 10, 7$$

$$\sqrt{7^2 - 40} = 3, \quad \frac{1}{2}(7 - 3) = 5, 2$$

Therefore,

$$(17 + \sqrt{40} + \sqrt{80} + \sqrt{200})^{1/2} = \sqrt{10} + \sqrt{5} + \sqrt{2}$$

Nārāyaṇa's Rules

For finding the square-root of a surd expression, Nārāyaṇa (1350) gives the following rules:

"The number of irrational terms in the square of a surd expression is equal to the sum of natural numbers: this is the usual rule. In the square of a single surd term, there is only a rational number. In the square of an expression consisting of two surd terms, there is one surd term together with a rational number; of three; three; of four; six; of five, ten; and in the square of an expression consisting of six surd terms, there will be as many as fifteen surd terms; so it should be known. In an expression having the number of surd terms

equal to the sum of the natural numbers, subtract from the square of the rational term a rational number equal to the sum of that number of surd numbers and then extract the square root of the remainder. Add and subtract this to the rational number and halve. The results are the two surd terms. If further terms remain to be operated upon, regard the greater of these two as a rational number and find the other terms (of the root) by proceeding as before. If the number of surd terms in any expression be not equal to the sum of the natural numbers, the (requisite) number should be made up by breaking up some of the terms and then the square-root should be extracted. If that is not possible, the problem is wrong[36]"

"Increase twice the number of surd terms (in a given expression) by one fourth and then extract the square-root. Subtract half from that. The residue will give the number of terms (the sum of which is to be subtracted from the square of the rational term)[37]".

"Or divide all the surd numbers (present in an expression) by four and arrange the quotients in the descending order. Divide the product of the two surds nearest to the first surd (in the series) by the latter. The square root of the quotient will be a surd term (in the root). Those two surds divided by this root will give another two surd terms (of the root). By these (three surds) divide next (three) terms of the series and the quotient will be another surd of the root. Again by these should be divided the other terms and the quotient is another surd; and so on. If now the square of the sum of surd numbers (in the root) be subtracted from the rational term (in the given expression) no remainder will be left. If it be not so (i.e., if a remainder is left), then the (given) square expression is a compound surd and it should be broken up into other surds by the rule of separation[38]".

Notes & References

1. *SiŚe (=Siddhānta-śekhara)*, xiv. 7.
2. *NBi (= Nārāyaṇa's Bijagaṇita)* I, R. 25; See also the commentaries of Gaṇeśa and Kṛṣṇa on the *Bijagaṇita* of Bhāskāra II.
3. *Gommaṭa-sāra* of Nemicandra, *Jīva Kāṇḍa, Gāthā* 170.
4. *BrSpSi (= Brāhma-sphuṭa-siddhānta)*, xviii. 38.
5. *GSS (= Gaṇita-sāra-saṅgraha)*, vii. $88^1/2$.
6. *SiŚe*, xiv. 8f
7. *BBi (= Bhāskara's Bījagaṇita)*, pp. 12f.
8. *NBi (Junction of a door)*, I, R. 25-30.
9. *BrSpSi*, xviii. 38.
10. *BrSpSi*, xviii. 39.
11. *SiŚe*, xiv. 9.
12. *SiŚe*, xiv. 11.

13. *BBi*, p. 13.
14. *NBi*, I, R. 31.
15. *BrSpSi*, xviii, 39.
16. *SiŚe*, xiv. 11.
17. *BBi*, p. 14.
18. *NBi*, I, R. 37-8.
19. *NBi*, I, R. 33-5
20. *BBi*, pp. 15, 16.
21. *NBi*, I, example on R. 33-5; also Ex. 18.
22. *BBi*, p. 15.
23. *NBi*, I, R. 36.
24. *BrSpSi*, xviii. 40
25. *SiŚe*, xiv. 12
26. *BBi*. pp. 17f.
27. *BBi*, p. 19.
28. *BBi*, pp. 19f.
29. *BBi*, p. 20.
30. *BBi*, pp. 20ff.
31. *BBi*, p. 23.
32. *BBi*, p. 23.
33. *BBi*, p. 24.
34. *BBi*, p. 24.
35. *BBi*, p. 24.
36. *NBi*, I, R. 41-5.
37. *NBi*, I, R. 50.
38. *NBi*, I, R. 46-9.

Indian Journal of History of Science, **28**(3), 1993

APPROXIMATE VALUES OF SURDS IN HINDU MATHEMATICS

BIBHUTIBHUSAN DATTA AND AWADHESH NARAYAN SINGH
(Revised by Kripa Shankar Shukla)
"Argara", Hussainganj Crossing, Behind Bata Shoe Co.,
Lucknow-226 001

(Received 24 December 1991)

As has been shown in an earlier article, the Hindu interest in the athematics of surds is very old. The ancient Hindus were interested not only in the operations of the surds but also in finding their approximate values. The present article gives an account of the methods used for this purpose.

The method to find approximate values of surds is found as early as the time of the *Śulba*. Thus, Baudhāyana (800 BC) states:

"Increase the measure (of which the *dvi-karaṇī* is to be found) by its third part, and again by the fourth part (of this third part) less by the thirty-fourth part of itself (i.e., of this fourth part). (The value thus obtained is called) the *saviśeṣa*" (approximate)[1].

That is to say, if *d* be the *dvi-karaṇī* of *a*, that is, if *d* be the side of a square whose area is double that of the square on *a* then we shall have:

$$d = a + \frac{a}{3} + \frac{a}{3.4} - \frac{a}{3.4.34}, \text{approx.}$$

whence, we get

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}, \text{ approx.}$$

Expressing in decimal fractions, we obtain $\sqrt{2} = 1.4142156....$ According to modern calculation, $\sqrt{2} = 1.414213....$ Thus, it is clear that the ancient Hindus attained a very remarkable degree of accuracy in calculating an approximate value of $\sqrt{2}$. There has been much speculation among modern writers about the method by which the Hindus arrived at this result[2]. The most recent hypothesis is that of Bibhutibhusan Datta[3]. It is based on a simple and elegant geometrical procedure quite in keeping with the spirit of the early Hindu geometry and hence seems to be a very plausible one. According to Nīlakaṇṭha (*c.* 1500)[4], Baudhāyana supposed the side of a square to be 12 units in length, so that its diagonal would be $\sqrt{2 \,.\, 12^2} = \sqrt{288}$ units. Now $\sqrt{288} = \sqrt{17^2 - 1}$

$$= 17 - \frac{1}{34}, \text{ nearly}$$

Therefore $12\sqrt{2} = 17 - \frac{1}{34}$, nearly.

Hence, $$\sqrt{2} = \frac{17}{12} - \frac{1}{12.34},$$

or $$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$$

Other notable approximate values occurring in the *Śulba* are[5]:

$$\sqrt{2} = \frac{7}{5},\ 1\frac{11}{25}, \qquad \sqrt{29} = 5\frac{7}{18},$$

$$\sqrt{5} = 2\frac{2}{7}, \qquad \sqrt{61} = 7\frac{5}{6}.$$

Probably it was also known that[6]

$$\sqrt{3} = 1 + \frac{2}{3} + \frac{1}{3.5} - \frac{1}{3.5.52}$$

In the early canonical works of the Jainas (500-300 BC)[7], we find applications of the formula

$$\sqrt{N} = \sqrt{a^2 + r} = a + \frac{r}{2a}$$

This formula has been applied consistently by the Jaina writers even up to the middle ages[8].

Bakhshālī Formula

In the Bakhshālī treatise on arithmetic (*c.* 200), we have the following rule for determining the approximate root (*śliṣṭa-mūla*, literally "nearest root") of a non-square number:

"In case of a non-square number, subtract the nearest square number; divide the remainder by twice (the root of that number). Divide half the square

of that (that is, the fraction just obtained) by the sum of the root and fraction and subtract. (This will be the approximate value of the root) less the square (of the last term)[9]".

This is to say,

$$\sqrt{N} = \sqrt{a^2 + r} = a + \frac{r}{2a} - \frac{(\frac{r}{2a})^2}{2\left(a + \frac{r}{2a}\right)}$$

approximately, the error being

$$\left[\frac{(\frac{r}{2a})^2}{2\left(a + \frac{r}{2a}\right)}\right]^2$$

Example from the work:

$$\sqrt{41} = 6 + \frac{5}{12} - \frac{(\frac{5}{12})^2}{2\left(6 + \frac{5}{12}\right)},$$

$$\sqrt{339009} = 579 + \frac{384}{579} - \frac{(384/579)^2}{2\,(579 + 384/579)}$$

In applying this approximate formula to concrete examples, the Bakhshali treatise exhibits an accurate method of calculating errors and an interesting process of reconciliation, the like of which are not met elsewhere[10].

Lalla's Formula

To find the square-root of a sexagesimal fraction Lalla gives the following rule:

"Find the square-root (of the integral part in minutes) by the method indicated before. Multiply by sixty the remainder plus unity and then add the seconds. The result divided by twice the root plus 2 will be the fractional part (of the square root in terms of seconds)[11]"

That is if $\alpha = \beta^2 + \epsilon$, then we shall have

$$\sqrt{\alpha' r''} = \beta' + \left\{ \frac{60(\epsilon + 1) + r}{2(\beta + 1)} \right\}''$$

in sexagesimal fractions. The same formula appears in the *Rājamṛgāṅka of* Bhojarāja and the *Karaṇa-Kutūhala* of Bhāskara II[12]. It is obviously based on the approximate formula:

$$\sqrt{a^2 + r} = a + \frac{r + 1}{2(a + 1)}$$

BRAHMAGUPTA'S FORMULA

Brahmagupta (628) says:

"The integer (in degrees), multiplied by its sexagesimal fraction (in minutes) and divided by thirty is (approximately) the square due to the fraction which is to be added to the square of the integer[13]".

That is, we have

$$(\alpha^\circ \beta')^2 = (\alpha + \frac{\beta}{60})^2$$

$$= \alpha^2 + \frac{\alpha\beta}{30} + (\frac{\beta}{60})^2$$

$$= \alpha^2 + \frac{\alpha\beta}{30}, \text{ nearly,}$$

neglecting $(\beta/60)^2$ as being very small.

From the above rule, we easily obtain a formula for finding the approximate value of a non-square number. For if x be a small fraction compared with a, we have

$$(a + x)^2 = a^2 + 2\,ax$$

Putting $2\,ax = r$, we get

$$x = \frac{r}{2a}$$

Hence

$$\sqrt{a^2 + r} = a + \frac{r}{2a}$$

Brahmagupta expressly states a formula very much akin to that found in the Bakhshali treatise. To find the square-root of the sum or the difference of the squares of two numbers, the larger of which has a fractional part, he gives the following rule:

"Divide the square of the given smaller number plus or minus the portion in the square of the other due to its fractional part by twice (the integral part of) the other (at one place) and (at a second place) by the latter plus or minus the quotient obtained at the other place. The (last) divisor being added or subtracted by the last quotient and halved gives the square-root of the sum or the difference of the tv squares. Or it is the other number plus or minus that quotient"[1].

That is, if a > b and ϵ, a small fraction, then

$$\sqrt{(a+\epsilon)^2 \pm b^2} = \frac{1}{2}\left\{2a \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a} \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a}}\right\} \qquad \text{(i)}$$

or

$$\sqrt{(a+\epsilon)^2 \pm b^2} = a \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a}} \qquad \text{(ii)}$$

The second formula gives an approximation by defect. The value

$$\sqrt{(a + \epsilon)^2 \pm b^2} = a \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a} \qquad \text{(iii)}$$

gives an approximation by excess. Taking the mean of (ii) and (iii), Brahmagupta finds the closer approximation given by (i).

On simplifying, we get from the formula (ii):

$$\sqrt{(a + \epsilon)^2 \pm b^2} = a \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a} \mp \frac{\left\{\frac{b^2 \pm (2a\epsilon + \epsilon^2)^2}{2a}\right\}^2}{2a + \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a}}$$

Putting $\epsilon = 0$, $b^2 = r$, we have the formula

$$\sqrt{a^2 \pm r} = a \pm \frac{r}{2a} \mp \frac{(\frac{r}{2a})^2}{2a \pm \frac{r}{2a}}$$

ŚRĪDHARA'S FORMULA

Śrīdhara (*c.* 750) gives the following rule for finding the approximate value of the square-root of a non-square number:

"Multiply the non-square number by some large square number; then take the square-root (of the product), neglecting the excess, and divide it by the root of the multiplier"[14]

$$\sqrt{N} = \frac{\sqrt{N m^2}}{m} = \frac{R}{m}, \quad \text{nearly}$$

where m is an arbitrary large number and R is the nearest integral root of Nm^2. Śrīdhara gives two illustrative examples:

$$\sqrt{1000} = \frac{\sqrt{1000 \times 10000}}{100} = \frac{3162}{100} = 31\frac{31}{50}$$

$$\sqrt{6250} = \frac{\sqrt{6250 \times 10000}}{100} = \frac{7905}{100} = 79\frac{1}{20}$$

There are found various other formulae based upon Śrīdhara's formula. Thus, Āryabhaṭa II (*c.* 950) gives[15]:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{ab \times 10000}}{b \times 100} = \frac{R}{b \times 100}$$

Śrīpati (1039) has[16]

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{ab \times m^2 \times 10000}}{b \times m \times 100} = \frac{R}{bm \times 100}$$

Bhāskara II (1150) states the formula[17]:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{abm^2}}{bm} = \frac{R}{bm}$$

Example from Bhāskara II:

$$\sqrt{\frac{169}{8}} = \frac{\sqrt{169 \times 8 \times 10000}}{800} = \frac{3677}{800} = 4\,\frac{477}{800}.$$

Munīśvara (1658) gives

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{ab \times 10{,}000{,}000{,}000{,}000{,}000}}{b \times 100{,}000{,}000} = \frac{R}{b \times 100{,}000{,}000}$$

Illustrative example from him[19]:

$$\sqrt{208} = \frac{\sqrt{2080{,}000{,}000{,}000{,}000{,}000}}{100{,}000{,}000} = \frac{1442220510}{100{,}000{,}000}$$

$$= 14\,\frac{4222051}{10000000}$$

NĀRĀYAṆ'S METHOD

Nārāyaṇa (1356) says:

"Obtain the roots (of a square-nature) having unity as the additive and the number whose square-root is to be determined (as the multiplier). Then the greater root divided by the lesser root will be the approximate value of the square-root[20]"

That is to say, to find the approximate value of the surd $\sqrt{N}$ we shall have to solve the quadratic indeterminate equation

$$N\,x^2 + 1 = y^2$$

If $x = \alpha$, $y = \beta$ be a solution of this equation, then, says Nārāyaṇa

$$\sqrt{N} = \frac{\beta}{\alpha}, \text{ approximately.}$$

In illustration of his method, Nārāyaṇa finds approximations to $\sqrt{10}$ and $\sqrt{1/5}$[21]. Since the roots of $10x^2 + 1 = y^2$ are (6, 19), (228, 721), (8658, 27379),......, we have

$$\sqrt{10} = \frac{19}{6}, \frac{721}{228}, \frac{27379}{8658}, \ldots$$

Again the values of (x, y) satisfying the equation

$$\sqrt{\frac{1}{5}} x^2 + 1 = y^2$$

are (20, 9), (360, 161), (6460, 2889), ... Therefore

$$\frac{1}{5} = \frac{9}{20}, \frac{161}{360}, \frac{2889}{6460}, \ldots$$

Jñānarāja's Method

Jñānarāja (1503) writes:

"Divide its square by the root of the nearest square number. The quotient together with that approximate root being halved will be a root more approximate than that. Values more and more accurate can certainly be found by proceeding in the same way repeatedly[22]".

In other words, if a^2 be the square number nearest to the non-square number N, so that $N = a^2 \pm r$, then the first approximate value (α_1) of $\sqrt{N}$ will be, says Jñānarāja,

$$\frac{1}{2}\left(a + \frac{N}{a}\right)$$

The next approximation will be

$$\frac{1}{2}\left\{\frac{1}{2}\left(a + \frac{N}{a}\right) + \frac{N}{\frac{1}{2}\left(a + \frac{N}{a}\right)}\right\}$$

and so on. The following illustrative examples are given:

$$\sqrt{8} = \frac{1}{2}\left(3 + \frac{8}{3}\right) = \frac{17}{6} = 2^\circ\ 50' \text{ approximately}$$

$$\sqrt{8} = \frac{1}{2}\left(\frac{17}{6} + \frac{8 \times 6}{17}\right) = \frac{577}{204} = 2^\circ\ 49'\ 42'', \text{ approximately}$$

$$= \frac{1}{2}(2^\circ\ .\ 50' + 2^\circ\ .\ 49'\ 42'')$$

$$= 2^\circ 49' 51'', \text{ approximately.}$$

$$\sqrt{20} = \frac{1}{2}\left(4 + \frac{20}{4}\right) = \frac{9}{2} = 4^\circ 30', \text{ approximately}$$

$$= \frac{1}{2}\left(\frac{9}{2} + \frac{20 \times 2}{9}\right) = \frac{161}{36} = 4^\circ 28' 20'', \text{ approximately}$$

$$= \frac{1}{2}(4^\circ 30' + 4^\circ 28' 20'')$$

$$= 4^\circ 29' 10'', \text{ approximately.}$$

FORMULA OF AN ANONYMOUS WRITER

In his commentary on the *Līlāvatī* of Bhāskara II, Gaṇeśa (1545) has quoted a rule from a "previous writer" (*Ādya*) for finding the approximate value of the square-root of a non-square number.

It runs as:

"The residue of the root together with unity is multiplied by 60 and divided by twice the root plus 1. The sixtieth part of the root added with this fraction is (the required approximate value of) the root"

The process implied is clearly this:

$$\sqrt{N} = \frac{\sqrt{3600N}}{60}$$

Now on finding the square-root of 3600 N by the ordinary method for it, suppose the root comes out to be b and the residue in excess r. Then according to the rule

$$\sqrt{N} = \frac{1}{60}\left\{b + \frac{60\,(r + 1)}{2\,(b + 1)}\right\}$$

in sexagesimal fractions. It is obviously based on the approximate formula

$$\sqrt{a^2 + r} = a + \frac{r + 1}{2(a + 1)}$$

KAMALĀKARA

Kamalākara (1658) mentions all the formulae for finding the approximate value of a surd from that of Śrīdhara onwards[23]. But he has always employed the formula of Lalla. Its *rationale* has been given by him to be as follows[24].

Suppose

$$\sqrt{b^2 + r} = b + \epsilon$$

where ϵ is a small quantity. Then

$$b^2 + r = b^2 + 2b\epsilon + \epsilon^2$$

or $\epsilon\,(2b + 2\epsilon) = r + \epsilon^2$

Therefore, $\epsilon = \dfrac{r + \epsilon^2}{2b + 2\epsilon}$

$$= \frac{r + 1}{2b + 2} \text{ approximately}$$

Hence, we have the approximate formula.

$$\sqrt{b^2 + r} = b + \frac{r + 1}{2b + 2}$$

or in sexagesimal fractions:

$$\sqrt{b^2 + r} = b + \frac{60(r + 1)}{2(b + 1)}$$

Examples:

$\sqrt{5} = 2°14'10''$

$\sqrt{10} = 3°9'44''12'''$

$\sqrt{468°5'} = 21°28'7''$.

By the repeated application of the method, Kamalākara also finds the fourth root of numbers, e.g.,

$$\sqrt[4]{10} = 1°46'41''36'''$$

References and Notes

1. *Baudhayānā Śulba,* i. 61-2; see also *Āpastamba Śulba,* i. 6; *Katyāyana Śulba,* ii. 13.

2. Thibaut, *Śulvasūtras,* pp. 13 ff; C. Muller, "Die Mathematik der Śulvasūtra," *Abhand, a.d. Math. Sem.d. Hamburg Univ.,* Bd. vii, 1929, pp. 173-204.

3. Bibhutibhusan Datta, *Śulba,* pp. 192ff.

4. Vide his commentary on the *Āryabhaṭiya,*ii. 4. His commentary has been published in the Trivandrum Sanskrit series (Nos. 101, 110 and 185).

5. Datta, *Śulba,* p. 205.

6. For an elegant method of getting this approximate value see Datta, *Śulba,* pp. 194 ff.

7. For instance, see *Jambūdvīpaprajñapti, Sūtra,* 3, 10-16; *Jīvābhigamasūtra, Sūtra* 82, 124; *Sūtrakṛtānga-sūtra, Sūtra* 12, etc.

8. See the commentaries of Siddhasena Gani (*c.* 550), Malaya-giri (*c.* 1200) and others.

9. This rule is not preserved in its entirety at any place in the surviving portion of the Bakhshali manuscript; but it can be easily restored from the cross-references, especially on the folios 56, recto and 57 verso. See Bibhutibhusan Datta, *Bakh. Math.,* pp. 11 ff.

10. Datta, *Bakh. Math.,*pp. 14ff.

11. *ŚiDVṛ* iii. 52

12. *Rājamṛgānka* vi. 26 (c-d) – 28 (a-b); *Karaṇa-Kutūhala, spaṣṭādhikāra,* vs. 14.

13. *BrSpSi,* xii. 62.

14. *Tris (= Triśatikā),*R. 46.

15. *MSi = Mahā-siddhanta),* xv. 55.

16. *SiŚe (= Siddhānta-Śekhara),* xiii. 36

17. *L (= Līlāvatī),* p. 34.

18. *PāSā (= Pāṭīsāra),* R. 117.

19. *PāSā,* R. 120.

20. *NBi (= Nārāyaṇa's Bījagan nita)* I, R. 88. Cf. Bibhutibhusan Datta, "Nārāyana's method for finding approximate value of a surd," *BCMS,* xxiii (1931), pp. 187-194.

21. *NBi,* I, Ex. 45

22. *Āsannamūlena hṛtāt svavargāl labdhena mūlaṁ sahitaṁ dvibhaktam* ।
Bhavedāsannapadaṁ tato'pi muhurmuhuḥ syāt sphuṭamūlamevam ॥
Sundara-Siddhānta, bījādhyāya, 12 (c-d)-13(a-b).

23. *SiTVi,* iii. 10-19.

24. *SiTVi,* xiv. 324 (com).